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Density perturbations arising from multiple field slow-roll inflation

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Abstract

In this paper we analyze scalar gravitational perturbations on a Robertson-Walker background in the presence of multiple scalar fields that take values on a (geometrically non-trivial) field manifold during slow-roll inflation. For this purpose modified and generalized slow-roll functions are introduced and their properties examined. These functions make it possible to estimate to what extent the gravitational potential decouples from the scalar field perturbations. The correlation function of the gravitational potential is calculated in an arbitrary state. We argue that using the vacuum state seems a reasonable assumption for those perturbations that can be observed in the CMBR. Various aspects are illustrated by examples with multiple scalar fields that take values on flat and curved manifolds.

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1 Introduction

As has been known for a long time, inflation [7, 12] offers a mechanism for the production of density perturbations, which are supposed to be the seeds for the formation of large scale structures in the universe. This mechanism is the magnification of microscopic quantum fluctuations in the scalar fields present during the inflationary epoch into macroscopic matter and metric perturbations. Also, since a part of the primordial spectrum of density perturbations is observed in the cosmic microwave background radiation (CMBR), this mechanism offers one of the most important ways of checking and constraining possible models of inflation, see e.g. [9].

The theory of the production of density perturbations in the case of a single real scalar field has been studied for a long time [3, 19, 10]. However, to realize inflation that leads to the observed density perturbations in a model without very unnatural values of the parameters, it is now thought that one needs more than one field. This is a strong motivation for hybrid inflation models [13] (more models can be found in [14]). Also, many high-energy theories contain a lot of scalar fields. The Higgs sector of the standard model consists of one physical particle, but in grand unification or supersymmetric models one expects many more scalars. Ultimately one would hope to be able to identify those fields that could act as inflatons. For all these reasons it is important to develop a theory for perturbations from multiple field inflation as well. Work in this direction has been done by several people. Using gauge invariant variables the authors of [23, 6] treated two field inflation. The fluid flow approach was extended to multiple fields in [14], while a more geometrical approach was used in [26, 22]; both methods assumed several slow-roll-like conditions on the potential. Using slow-roll approximations for both the background and the perturbation equations the authors of [20, 24] were able to find expressions for the metric perturbations in multiple field inflation.

In this paper we generalize the slow-roll parameters for a single background field to the multiple field case in a systematic way. We can then give a clear quantification of the relative importance of terms in the equations obtained by extending the single field density perturbation calculations by Mukhanov, Feldman and Brandenberger [19] to multiple fields. In our definition of the slow-roll functions we do not implicitly assume slow roll to be valid. As a consequence, these slow-roll functions are expressed in terms of derivatives of the field velocity and the Hubble parameter, but not the potential as the conventional slow-roll parameters are. A big advantage of this is that these slow-roll functions can be identified in all kinds of equations that are still exact, which is the reason why we can estimate the relative importance of various terms and make a well-motivated decision about neglecting some of them. In the case of multiple scalar field inflation it is very convenient to think in terms of vectors: if the fields are local coordinates on a curved manifold, their derivatives and their fluctuations can be interpreted as covariant vectors in the tangent bundle of the manifold. Since the fluctuations are assumed to be small, a linearization procedure can be used to obtain equations for the perturbations. In these equations a prominent role is played by the scalar field velocity. Since in single scalar field inflation the only direction is parallel to the field velocity, it is to be expected that the parallel field perturbations can be absorbed in the gravitational perturbations, as happens in the single field theory. Using the modified definitions of the slow-roll functions we can investigate to what extent this is correct.

In the CMBR spectrum we can observe correlations in the temperature distribution.

They are assumed to be due to gravitational perturbations that are of a quantum origin at the beginning of inflation. To calculate these correlations we therefore need to address the question of what is the relevant state at the initial stages of inflation. We argue that the conventionally used vacuum state seems a good assumption. We investigate the effect on the correlator in the (probably unrealistic) case that the true state at the beginning of inflation is thermal with a temperature of the order of the Planck scale. For most of this work we restrict ourselves to the calculation of the correlation function near the end of inflation. However, to show that our results are consistent with other results in the literature, the correlation function is also evaluated at the time of recombination, ignoring possible problems during (p)reheating.

During inflation there is a relatively sharp transition in the behaviour of a fluctuation when the corresponding wavelength 'passes through the horizon' (this is discussed in detail in this paper). This moment of horizon crossing can be used to identify a certain scale k. The smallest scale (or largest wavelength) that can be observed in the CMBR is the one that is reentering the horizon at this very moment, indicated by k_0. Assuming that the fluctuations do not change once they have passed outside the horizon,[1] we can observe the undisturbed inflationary perturbation spectrum up to the scale that reentered the horizon at the time of recombination when the CMBR was formed, which has $\ln(k_{\mathrm{rec}}/k_0) \approx 3.5$. On the other hand, larger scales already reentered the horizon before this time, and so on these scales the spectrum has been influenced by physical processes taking place long after inflation. The largest scale expected to be measured with the Planck satellite has $\ln(k/k_0) \approx 7$ (corresponding with multipole $l = 2000$). An important quantity in our discussions is the number of e-folds N_k that occur after a certain scale k crosses the horizon during inflation, until the end of inflation. As is derived in e.g. equation (5.16) of [10], N_k depends logarithmically on k, taking a value of about 60 for k_0. This number has only a logarithmic dependence on model-dependent quantities like the reheating temperature. Hence the observationally important scales cross the horizon about 50 to 60 e-folds before the end of inflation.

Apart from this introduction and the conclusions at the end, the paper is structured as follows. In section 2 the theory of scalar perturbations with a gravitational potential coupled to a single scalar field is reviewed, following the methods discussed in [19]. This allows us to introduce various relevant concepts. In addition, this section also makes it possible to compare the single and multiple scalar field situations in the following sections.

The generalization to multiple scalar field inflation is developed in section 3. As these scalar fields parameterize a possibly curved manifold, it is necessary and convenient to introduce some geometrical tools in subsection 3.1. Two examples of these are the inner product and the covariant derivatives associated with the metric of the manifold. Another concept introduced here is the projection on directions parallel and perpendicular to the background field velocity, which plays an important role in the decoupling of the gravitational perturbations from the independent scalar field fluctuations. In the next subsection we explain why the single field method does not lead to an immediate decoupling of the gravitational perturbations in the multiple field case. To be able to compare the relative size of terms in the equations we define modified versions of the slow-roll functions that

[1] In [29] it was proved that this is always true for the so-called adiabatic perturbations, but it need not be true for the so-called isocurvature perturbations, see [5]. A full treatment of this complication is beyond the scope of this paper.

can be used in multiple field inflation in subsection 3.3 and derive various useful properties for them. In subsection 3.4 we derive the equation of motion for the gravitational potential in terms of these slow-roll functions and show that the term coupling this equation to the perpendicular field perturbations is small, so that also in the multiple field case there is an effective decoupling. The equation of motion for the perpendicular scalar field perturbations is derived in the appendix using similar methods. A discussion of the solution for the decoupled gravitational potential concludes this subsection.

Section 4 is devoted to the computation of the quantum correlation function of the gravitational potential. We argue why taking a vacuum state at the beginning of inflation to evaluate this correlator seems a good approximation.

In section 5 we analyze the behaviour of the background scalar fields during slow-roll inflation in various multiple field cases. In subsection 5.1 two examples of a quadratic potential on a flat manifold are considered: with equal and with different masses. Next, we turn to the generalized situation of a curved manifold with arbitrary potential in subsection 5.2. To illustrate some aspects we take the curved manifold to be the sphere with embedding coordinates. In each of these examples we compute the model dependent factor that appears in the expression for the correlation function of the gravitational potential.

2 Scalar perturbations

2.1 Linearized gravitational perturbations

We can divide general perturbations in the universe in the following two types: metric perturbations and matter perturbations. Of course they are related by the Einstein equations. On the other hand we can divide both matter and metric perturbations in three different classes: scalar, vector, and tensor perturbations [2, 28, 19], depending on how they transform under spatial transformations of the background metric. In this paper we consider only scalar perturbations since they are the main cause of the fluctuations in the CMBR. We assume throughout this paper that all perturbations are small, as on the one hand they presumably originate from quantum perturbations, while on the other hand the fluctuations in the CMBR that we observe are tiny. In particular this means that we linearize all equations with respect to the perturbations. This section is essentially a review of [19].

The Robertson-Walker metric for a spatially flat background combined with scalar metric perturbations may be written as

$$g_{\mu\nu}^{\text{full}}(\eta,\mathbf{x}) = g_{\mu\nu}(\eta) + \delta g_{\mu\nu}(\eta,\mathbf{x}) = a^2 \begin{pmatrix} -1 & 0 \\ 0 & \delta_{ij} \end{pmatrix} + a^2 \begin{pmatrix} -2\Phi & B_{,j} \\ B_{,i} & -2\Psi\delta_{ij} + 2E_{,ij} \end{pmatrix}. \quad (1)$$

We take a flat background with scale factor $a(\eta)$, since we are going to apply our formulae to the universe during and after inflation, when it has already been inflated to complete flatness. Φ, Ψ, B, E are four scalar functions of spatial coordinates $\mathbf{x}$ and conformal time η which together describe the metric perturbations. One often refers to Φ as the Newtonian potential, as the 00-component of the metric in a weak field approximation can be identified with the potential of Newtonian gravity. The conformal time η is related to the comoving time t by $\mathrm{d}t = a\mathrm{d}\eta$. The advantage of using conformal time is that then the scale factor a is an overall factor of the full metric, not just of the spatial part. We take the scale factor to have dimension of length and therefore η and $\mathbf{x}$ are dimensionless. Differentiation with respect to conformal and comoving time are denoted by $' \equiv \partial_\eta$ and $\dot{} \equiv \partial_t$, respectively. The conformal Hubble parameter $\mathcal{H}$ is defined as $\mathcal{H} \equiv a'/a$. It is related to the comoving Hubble parameter $H \equiv \dot{a}/a$ by $\mathcal{H} = aH$. The notation $_{,i}$ denotes a derivative with respect to the spatial coordinate x^i.

There is a problem with the interpretation of the metric perturbations because it is difficult to separate the physical metric perturbations from the ones that can be gauged away by a coordinate transformation. Of course the final, physical results do not depend on the choice of coordinates. However, it is often convenient if intermediate results are also independent of the gauge chosen. To this end so-called gauge-invariant quantities are introduced [2, 19], which are defined to be gauge invariant with respect to infinitesimal coordinate redefinitions. In this approach one defines gauge-invariant metric and matter quantities and uses those in the full system of metric and matter perturbations.[2] They are:

$$\Phi^{(gi)} = \Phi + \frac{1}{a}\left[(B - E')a\right]', \quad \Psi^{(gi)} = \Psi - \mathcal{H}(B - E'), \quad \delta q^{(gi)} = \delta q + q'(B - E'). \quad (2)$$

Here we have separated an arbitrary scalar (matter) quantity $q_{\text{full}}(\eta,\mathbf{x})$ into a homogeneous background part $q(\eta)$ and a perturbation $\delta q(\eta,\mathbf{x}) = q_{\text{full}}(\eta,\mathbf{x}) - q(\eta)$, just as we did for the

[2] The main alternative is the fluid flow approach [10, 14], where one makes certain assumptions about the matter content of the universe, which enables one to eliminate the metric perturbations and derive equations for the matter perturbations in closed form.

metric. As can be seen from (2), working with these gauge-invariant quantities is equivalent to choosing the longitudinal gauge in which $B = E = 0$. As we only employ the longitudinal gauge in this paper, we omit the (gi) labels without risk of confusion.

For most of the treatment in this paper it is irrelevant whether scalar perturbations like $\delta q(\eta, \mathbf{x})$ and $\Phi(\eta, \mathbf{x})$ are classical or quantum objects. This is because we will linearize in those quantities so that the quantum nature (such as variables that do not commute) does not play a role. Hence we may derive and manipulate the equations as if all quantities were classical. Only when we are computing the correlator of the Newtonian potential $\langle\Phi\Phi\rangle$ in section 4 do we have to take the quantum nature of the perturbations into account.

We now treat the dynamics of the universe with scalar perturbations. The background Einstein equations read

$$\mathcal{H}^2 = -\frac{1}{3}\kappa^2 a^2 T^0_{\ 0}, \qquad \left(2\mathcal{H}' + \mathcal{H}^2\right)\delta^i_j = -\kappa^2 a^2 T^i_{\ j}, \tag{3}$$

and $T^0_{\ i} = 0$. Expanding the Einstein equations to first order in the perturbations gives

$$\begin{aligned} -3\mathcal{H}(\mathcal{H}\Phi + \Psi') + \Delta\Psi &= -\frac{1}{2}\kappa^2 a^2 \delta T^0_{\ 0}, & (4)\\ \left(\mathcal{H}\Phi + \Psi'\right)_{,i} &= -\frac{1}{2}\kappa^2 a^2 \delta T^0_{\ i}, & (5)\\ \left[(2\mathcal{H}' + \mathcal{H}^2)\Phi + \mathcal{H}\Phi' + \Psi'' + 2\mathcal{H}\Psi' + \frac{1}{2}\Delta(\Phi - \Psi)\right]\delta^i_j - \frac{1}{2}\delta^{ik}(\Phi - \Psi)_{,kj} &= \frac{1}{2}\kappa^2 a^2 \delta T^i_{\ j}, & (6) \end{aligned}$$

with $\kappa^2 \equiv 8\pi G = 8\pi/M_P^2$ and $\Delta = \sum_i \partial_i\partial_i$. Later on we often switch to complex Fourier modes $f_{\mathbf{k}}(\eta)$, defined by

$$f(\eta, \mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3\mathbf{k} \left(f_{\mathbf{k}}(\eta) e^{-i\mathbf{k}\mathbf{x}} + f^*_{\mathbf{k}}(\eta) e^{i\mathbf{k}\mathbf{x}}\right), \tag{7}$$

where f is any real quantity that depends on both time and space coordinates, e.g. $\Phi(\eta, \mathbf{x})$. After this switch equations for $f(\eta, \mathbf{x})$ become equations for $f_{\mathbf{k}}(\eta)$ and the spatial Laplacian $-\Delta$ is replaced by $k^2 = |\mathbf{k}|^2$.

The complicated system of Einstein equations is simplified considerably when the matter is described by a scalar field theory. For a scalar field theory with an arbitrary number of fields one can easily verify that $\delta T^i_{\ j} \propto \delta^i_j$ to first order in the perturbations, while $\delta T^0_{\ i}$ can be written as $\delta T^0_{\ i} = \delta F_{,i}$ to the same order. Here δF is a scalar function of the fields and their perturbations which depends on the scalar field theory. A very important simplification then follows from an argument by Mukhanov et al. [19], who show that one can take $\Psi = \Phi$ if $T^i_{\ j} \propto \delta^i_j$.

By taking a normalized trace of the (ij)-components and subtracting the (00)-component of the background Einstein equations (3), one obtains the following equation:

$$\mathcal{H}^2 - \mathcal{H}' = \frac{1}{2}\kappa^2 a^2 \left(\frac{1}{3}T^i_{\ i} - T^0_{\ 0}\right). \tag{8}$$

A similar procedure with the perturbed components of the Einstein equations (4) and (6), setting $\Psi = \Phi$, gives the equation of motion for the Newtonian potential Φ

$$\Phi'' + 6\mathcal{H}\Phi' + 2(\mathcal{H}' + 2\mathcal{H}^2)\Phi - \Delta\Phi = \frac{1}{2}\kappa^2 a^2 \left(\frac{1}{3}\delta T^i_{\ i} + \delta T^0_{\ 0}\right). \tag{9}$$

With a suitable choice for the Newtonian potential Φ to eliminate the integration constant the perturbed $(0i)$-component (5) of the Einstein equations can be integrated using $\delta T^0_i = \delta F_{,i}$:

$$\Phi' + \mathcal{H}\Phi = -\frac{1}{2}\kappa^2 a^2 \delta F. \tag{10}$$

In the next subsection we apply these equations to the case of a scalar field theory with a single field, and in section 3.2 to the multiple field case.

2.2 Scalar perturbations due to a single scalar field

Now we consider the case of a single real scalar field, both to review the method of Mukhanov et al. [19] and to be able to compare it with the multiple scalar field case when we treat the latter in section 3. From the Lagrangean

$$\mathcal{L} = \sqrt{-g}\left(-\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi)\right) \tag{11}$$

we obtain the equation of motion for the scalar field,

$$D^\mu\partial_\mu\phi - V_{,\phi} = 0, \tag{12}$$

where D_μ is the covariant space-time derivative, and the energy-momentum tensor

$$T^\mu_\nu = \partial^\mu\phi\partial_\nu\phi - \delta^\mu_\nu\left(\frac{1}{2}\partial^\lambda\phi\partial_\lambda\phi + V(\phi)\right). \tag{13}$$

In these three equations, ϕ denotes the total field $\phi_{\rm full}$. Now we separate the background from the perturbations, as defined below (2), so that in the remaining equations ϕ denotes the background part of the field. The integrated $(0i)$-Einstein equation (10),

$$\Phi' + \mathcal{H}\Phi = \frac{1}{2}\kappa^2\phi'\delta\phi, \tag{14}$$

and the background equation of motion for the scalar field,

$$\phi'' + 2\mathcal{H}\phi' + a^2 V_{,\phi} = 0, \tag{15}$$

can be used to eliminate the fluctuation $\delta\phi$ and the first derivative of the potential $V_{,\phi}$ from the equation of motion (9) for the Newtonian potential Φ,

$$\Phi'' + 6\mathcal{H}\Phi' + 2(\mathcal{H}' + 2\mathcal{H}^2)\Phi - \Delta\Phi = -\kappa^2 a^2 V_{,\phi}\delta\phi. \tag{16}$$

Then this last equation takes the form of a homogeneous differential equation:

$$\Phi'' + 2\left(\mathcal{H} - \frac{\phi''}{\phi'}\right)\Phi' + 2\left(\mathcal{H}' - \mathcal{H}\frac{\phi''}{\phi'}\right)\Phi - \Delta\Phi = 0. \tag{17}$$

Using equation (8),

$$\mathcal{H}^2 - \mathcal{H}' = \frac{1}{2}\kappa^2\phi'^2, \tag{18}$$

equation (17) can be written in terms of $u \equiv \frac{a}{\kappa^3 \phi'} \Phi$ as

$$u'' - \frac{\theta''}{\theta} u - \Delta u = 0, \qquad \text{with} \qquad \theta \equiv \frac{\mathcal{H}}{a\phi'}. \tag{19}$$

The factor κ^{-3} in the definition of u has been introduced to give it mass dimension one. We can use (18) and the relation $\mathcal{H} = aH$ to obtain expressions for $u = \Phi/(\kappa^2 \sqrt{-2\dot{H}})$ and $\theta = \kappa H/(a\sqrt{-2\dot{H}})$ that do not contain the scalar field ϕ, but only quantities that are well-defined also after inflation. Equation (19) for the variable u is very important because it can be used throughout the evolution of the universe: during scalar field, radiation, and matter domination [19] (only during matter domination an extra factor containing the sound velocity has to be added).

By varying (12) to first order in the perturbations we find the equation of motion for the scalar field fluctuations:

$$\left(\partial_\eta^2 + 2\mathcal{H}\partial_\eta - \Delta + a^2 V_{,\phi\phi}\right) \delta\phi = 4\phi'\Phi' - 2a^2 V_{,\phi}\Phi. \tag{20}$$

Notice that we did not need to use this equation in our derivation of a homogeneous equation for Φ (17) or u (19). It was not needed, because the equation of motion of the scalar field can be derived from the constraint that the energy-momentum tensor is divergenceless, $D_\mu T^{\mu\nu} = 0$, and is therefore not an independent equation. This is closely related to the fact that we could solve for $\delta\phi$ by dividing the integrated $(0i)$-Einstein equation (14) by the velocity ϕ'. In the case of more fields this constraint can no longer reproduce all equations of motion, so we expect to need the equations of motion for the field perturbations in that case.

3 Inflation with multiple scalar fields

3.1 Geometrical concepts

We now turn to the multiple scalar field case, where the scalars $\boldsymbol{\phi} = (\phi^a)$ can be interpreted as the coordinates of a real manifold $\mathcal{M}$ on which a metric $\mathbf{G}$ is defined. To make optimal use of the geometrical structure of this manifold in our discussion of the dynamics of scalar fields and their perturbations, we need to introduce some geometrical concepts. From the components of metric G_{ab} the metric-connection Γ^a_{bc} is obtained using the metric postulate. The definition of the manifold $\mathcal{M}$ is coordinate independent, therefore the description of this manifold is invariant under non-singular local coordinate transformations

$$\phi^a \longrightarrow \tilde{\phi}^a = X^a(\phi). \tag{21}$$

In our treatment of scalar perturbations due to multiple scalar fields we heavily rely on the concept of tangent vectors. A vector $\mathbf{A} = (A^a)$ is called a vector in the tangent space $T_p\mathcal{M}$ at a point $p \in \mathcal{M}$ if it transforms as

$$A^a \longrightarrow \tilde{A}^a = X^a_{\ b}(\phi) A^b, \qquad\qquad X^a_{\ b}(\phi) = X^a_{\ ,b}(\phi), \tag{22}$$

where the comma denotes differentiation with respect to local coordinates. A simple example of a (tangent) vector is the differential $\mathrm{d}\boldsymbol{\phi}$. The cotangent space is the dual of the tangent space. Its elements are linear operators on the tangent space

$$\begin{array}{cccc} {}^*\mathbf{C}: & T_p\mathcal{M} & \longrightarrow & \mathbb{R} \\ & \mathbf{A} & \mapsto & C_a A^a \end{array} \tag{23}$$

As $C_a A^a$ is a scalar object, the cotangent vector ${}^*\mathbf{C}$ transforms as

$$C_a \longrightarrow \tilde{C}_a = C_b (X^{-1})^b_{\ a}. \tag{24}$$

The metric G_{ab} can be used to construct a cotangent vector $(\mathbf{A}^\dagger)_a \equiv A^b G_{ba}$ from the tangent vector $\mathbf{A}$. Using index-free notation this reads $\mathbf{A}^\dagger = \mathbf{A}^T\mathbf{G}$. The notion of (co)tangent vectors defined at a point $p \in \mathcal{M}$ can be extended over the whole manifold $\mathcal{M}$ by interpreting them as sections of the (co)tangent bundle.

Using the metric $\mathbf{G}$ we introduce an inner product of two vectors $\mathbf{A}$ and $\mathbf{B}$ on the tangent space of the manifold and the corresponding norm

$$\mathbf{A}\cdot\mathbf{B} \equiv \mathbf{A}^\dagger\mathbf{B} = \mathbf{A}^T\mathbf{G}\mathbf{B} = A^a G_{ab} B^b, \qquad\qquad |\mathbf{A}| \equiv \sqrt{\mathbf{A}\cdot\mathbf{A}}. \tag{25}$$

The Hermitean conjugate $\mathbf{L}^\dagger$ of a linear operator $\mathbf{L} : T_p\mathcal{M} \longrightarrow T_p\mathcal{M}$ with respect to this inner product is defined by

$$\mathbf{B}\cdot(\mathbf{L}^\dagger\mathbf{A}) \equiv (\mathbf{L}\mathbf{B})\cdot\mathbf{A}, \tag{26}$$

so that $\mathbf{L}^\dagger = \mathbf{G}^{-1}\mathbf{L}^T\mathbf{G}$. A Hermitean operator $\mathbf{H}$ satisfies $\mathbf{H}^\dagger = \mathbf{H}$. An important example of Hermitean operators are the projection operators. Apart from being Hermitean, a projection operator $\mathbf{P}$ is idempotent: $\mathbf{P}^2 = \mathbf{P}$.

To complete our discussion on the geometry of $\mathcal{M}$ we introduce different types of derivatives. In the first place we have the covariant derivative on the manifold, denoted by $_{;a}$ or ∇_a, which acts in the usual way, i.e.

$$A^a_{\ ;b} = \nabla_b A^a \equiv A^a_{\ ,b} + \Gamma^a_{bc} A^c \tag{27}$$

on a vector A^a, and $V_{;a} = \nabla_a V = V_{,a}$ on a scalar V. It is convenient to also introduce index-free notation for (covariant) derivatives. On a scalar function V (e.g. the potential), the derivative $\boldsymbol{\partial}$ and the covariant derivative $\boldsymbol{\nabla}$ are equal

$$(\boldsymbol{\nabla} V)_a = (\boldsymbol{\partial} V)_a \equiv V_{,a}. \tag{28}$$

Since we represent $\mathrm{d}\boldsymbol{\phi}$ as a standing vector, $\boldsymbol{\nabla}$ and $\boldsymbol{\partial}$ are naturally lying vectors and therefore $\boldsymbol{\nabla}^T$ and $\boldsymbol{\partial}^T$ are standing vectors. The second covariant derivative of a scalar function V is a matrix with two lower indices:

$$(\boldsymbol{\nabla}^T \boldsymbol{\nabla} V)_{ab} = \nabla_a \nabla_b V. \tag{29}$$

Of course, the same holds good for ordinary derivatives $\boldsymbol{\partial}$.

The curvature tensor of the manifold can be introduced using tangent vectors $\mathbf{B}, \mathbf{C}, \mathbf{D}$:

$$[\mathbf{R}(\mathbf{B}, \mathbf{C})\mathbf{D}]^a \equiv R^a_{\ bcd}\, B^b C^c\, D^d \equiv \left(\Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^e_{bd}\Gamma^a_{ce} - \Gamma^e_{bc}\Gamma^a_{de}\right) B^b C^c\, D^d. \tag{30}$$

One should realize that for later notational convenience we do not use the standard definition as made for example in [21]: our $\mathbf{R}(\mathbf{B}, \mathbf{C})\mathbf{D}$ is conventionally denoted by $\mathbf{R}(\mathbf{C}, \mathbf{D}, \mathbf{B})$.

Next we discuss how spacetime derivatives act on spacetime dependent tangent vectors and their derivatives. Purely spacetime covariant derivatives are denoted by D_μ and are defined in the usual way. The covariant derivative $\mathcal{D}_\mu$ on a vector $\mathbf{A}$ of the tangent bundle is defined in components as

$$\mathcal{D}_\mu A^a \equiv \partial_\mu A^a + \Gamma^a_{bc} \partial_\mu \phi^b A^c, \tag{31}$$

while $\mathcal{D}_\mu$ acting on a scalar is simply equal to ∂_μ.

After the introduction of this standard geometrical machinery, we now develop some concepts to describe a time dependent scalar field background. They are used when we consider multiple field inflation. Consider a curve $\boldsymbol{\phi}(t)$ on a manifold $\mathcal{M}$, parameterized by a real variable t. In later sections, when we describe the time evolution of coupled systems consisting of multiple scalar fields and gravity, this variable t is interpreted as comoving time. Along this curve the nth derivative vector can be defined by

$$\boldsymbol{\phi}^{(1)} \equiv \dot{\boldsymbol{\phi}} = \frac{\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}t} \qquad \text{and} \qquad \boldsymbol{\phi}^{(n)} \equiv \mathcal{D}_t^{(n-1)} \dot{\boldsymbol{\phi}} \quad \text{for} \quad n \geq 2. \tag{32}$$

In applications in later sections $\boldsymbol{\phi}^{(1)}$ and $\boldsymbol{\phi}^{(2)}$ represent the velocity and acceleration of the background scalar fields, respectively. In general the vectors $\boldsymbol{\phi}^{(1)}, \boldsymbol{\phi}^{(2)}, \ldots$ do not point in the same direction. From these vectors a set of orthonormal unit vectors is obtained by using the Gram-Schmidt orthogonalization process. The first unit vector $\mathbf{e}_1$ is given by the direction of $\boldsymbol{\phi}^{(1)}$. The second unit vector $\mathbf{e}_2$ is determined by that part of $\boldsymbol{\phi}^{(2)}$ that is perpendicular to $\mathbf{e}_1$, and so on. To obtain the direction of $\boldsymbol{\phi}^{(2)}$ perpendicular to $\mathbf{e}_1$, we use the projection operators $\mathbf{P}_1$ and $\mathbf{P}_1^\perp$ that project on subspaces parallel and perpendicular to $\boldsymbol{\phi}^{(1)}$, respectively, and require that $\mathbf{e}_2$ is proportional to $\mathbf{P}_1^\perp \boldsymbol{\phi}^{(2)}$.

These definitions can be extended to $\mathbf{e}_n, \mathbf{P}_n$, etc., for any n. The unit vector $\mathbf{e}_n$ points in the direction of $\boldsymbol{\phi}^{(n)}$ perpendicular to the first $n-1$ unit vectors $\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}$. The operator $\mathbf{P}_n$ projects on $\mathbf{e}_n$ and $\mathbf{P}_n^\perp$ projects on the subspace which is perpendicular to $\mathbf{e}_1, \ldots, \mathbf{e}_n$. This subspace is also perpendicular to the derivative vectors $\boldsymbol{\phi}^{(1)}, \ldots, \boldsymbol{\phi}^{(n)}$. To obtain all

these objects at once, we let $\mathbf{P}_0^\perp = \mathbb{1}$ be the identity and define the mutually orthonormal unit vectors $\mathbf{e}_n$ (for $n = 1, 2, \dots$) from $\boldsymbol{\phi}^{(n)}$ and the projection operator $\mathbf{P}_{n-1}^\perp$ by

$$\phi_n^{(n)} = |\mathbf{P}_{n-1}^\perp \boldsymbol{\phi}^{(n)}|, \qquad \mathbf{e}_n = \frac{\mathbf{P}_{n-1}^\perp \boldsymbol{\phi}^{(n)}}{\phi_n^{(n)}}, \qquad \mathbf{P}_n = \mathbf{e}_n \mathbf{e}_n^\dagger, \qquad \mathbf{P}_n^\perp = \mathbb{1} - \sum_{q=1}^{n} \mathbf{P}_q. \tag{33}$$

By construction the vector $\boldsymbol{\phi}^{(n)}$ can be expanded in these unit vectors as

$$\boldsymbol{\phi}^{(n)} = (\mathbf{P}_1 + \ldots + \mathbf{P}_n)\, \boldsymbol{\phi}^{(n)} = \sum_{p=1}^{n} \phi_p^{(n)} \mathbf{e}_p, \qquad \phi_p^{(n)} = \mathbf{e}_p \cdot \boldsymbol{\phi}^{(n)}. \tag{34}$$

In particular, we have that $\phi_n^{(n)} = \mathbf{e}_n \cdot \boldsymbol{\phi}^{(n)} = |\mathbf{P}_{n-1}^\perp \boldsymbol{\phi}^{(n)}|$. As the projection operators $\mathbf{P}_1$ and $\mathbf{P}_1^\perp$ will occur frequently in later sections, we introduce the short-hand notation:

$$\mathbf{P}^\parallel = \mathbf{P}_1 = \frac{\dot{\boldsymbol{\phi}} \dot{\boldsymbol{\phi}}^\dagger}{|\dot{\boldsymbol{\phi}}|^2}, \qquad \mathbf{P}^\perp = \mathbf{P}_1^\perp = \mathbb{1} - \mathbf{P}^\parallel. \tag{35}$$

In terms of these two operators we can write a general vector and matrix as follows:

$$\mathbf{A} = \mathbf{A}^\parallel + \mathbf{A}^\perp, \qquad \mathbf{M} = \mathbf{M}^{\parallel\parallel} + \mathbf{M}^{\parallel\perp} + \mathbf{M}^{\perp\parallel} + \mathbf{M}^{\perp\perp}, \tag{36}$$

with $\mathbf{A}^\parallel = \mathbf{P}^\parallel \mathbf{A}$ and $\mathbf{M}^{\parallel\,\parallel} \equiv \mathbf{P}^\parallel \mathbf{M} \mathbf{P}^\parallel$, etc. Notice that because of the hermiticity of the projection operator $\mathbf{A}^\perp \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{B}^\perp = \mathbf{A}^\perp \cdot \mathbf{B}^\perp$.

3.2 Multiple scalar fields and gravitational perturbations

We now consider scalar fields $\boldsymbol{\phi}$ that are the local coordinates of a manifold $\mathcal{M}$, using the geometrical concepts introduced in the previous section. The space-time derivative of the background field $\partial_\mu \boldsymbol{\phi}$ transforms as a vector, even though the fields $\boldsymbol{\phi}$ in general do not, as they are coordinates on a manifold. Also the field perturbation $\boldsymbol{\delta\phi}$ and its gauge-invariant form defined by (2) transform as vectors. Only the covariance with respect to the coordinate transformations (21) of the target space $\mathcal{M}$ is manifest in our treatment, because we use a flat Robertson-Walker background and work to first order in the perturbations.

The Lagrangean for the scalar field theory with potential V on the manifold $\mathcal{M}$ can be written as

$$\mathcal{L}_\mathcal{M} = \sqrt{-g} \left(-\frac{1}{2} \partial^\mu \boldsymbol{\phi} \cdot \partial_\mu \boldsymbol{\phi} - V \right). \tag{37}$$

The equations of motion for the scalars are given by

$$g^{\mu\nu} \left(\mathcal{D}_\mu \delta_\nu^\lambda - \Gamma_{\mu\nu}^\lambda \right) \partial_\lambda \boldsymbol{\phi} - \mathbf{G}^{-1} \boldsymbol{\nabla}^T V = 0, \tag{38}$$

and the energy-momentum tensor is

$$T_\nu^\mu = \partial^\mu \boldsymbol{\phi} \cdot \partial_\nu \boldsymbol{\phi} - \delta_\nu^\mu \left(\frac{1}{2} \partial^\lambda \boldsymbol{\phi} \cdot \partial_\lambda \boldsymbol{\phi} + V \right). \tag{39}$$

In these three equations, $\boldsymbol{\phi}$ denotes the total field $\boldsymbol{\phi}_{\text{full}}$. Now we separate the background from the perturbations, as defined below (2), so that from now on $\boldsymbol{\phi}$ will always denote the

background part of the field. In this case of multiple scalar fields the equation of motion (9) of the Newtonian potential Φ reads to first order in the perturbations

$$\Phi'' + 6\mathcal{H}\Phi' + 2(\mathcal{H}' + 2\mathcal{H}^2)\Phi - \Delta\Phi = -\kappa^2 a^2 (\boldsymbol{\nabla} V\, \boldsymbol{\delta\phi}). \tag{40}$$

The integrated $(0i)$-component (10) of the Einstein equations takes the form

$$\Phi' + \mathcal{H}\Phi = \frac{1}{2}\kappa^2 \boldsymbol{\phi}' \cdot \boldsymbol{\delta\phi}. \tag{41}$$

In this case it is not possible to construct the analogue of the single field homogeneous equation of motion (17) for the Newtonian potential Φ, because (41) no longer contains a simple multiplication of two scalars, but an inner product of two vectors. Therefore, one cannot extract an explicit expression for $\boldsymbol{\delta\phi}$, as was the case in equation (14) for the single field situation. To overcome this difficulty, we divide the field perturbation $\boldsymbol{\delta\phi}$ in a part that is parallel to the velocity field $\boldsymbol{\phi}'$ and a part that is perpendicular, using the projection operators (35) defined in the previous section.[3] Here we use that for the projection operators $\mathbf{P}^{\parallel}$ and $\mathbf{P}^{\perp}$ there is no difference between using comoving time or conformal time: they only depend on the direction of $\dot{\boldsymbol{\phi}}$, which is the same as the direction of $\boldsymbol{\phi}'$. Once we have separated the fields in this way, the parallel part $\boldsymbol{\delta\phi}^{\parallel}$ can be eliminated in a way analogous to the single field case.

Using the integrated $(0i)$-component of the Einstein equations (41) together with the background equation of motion for the scalar fields,

$$\mathcal{D}_\eta \boldsymbol{\phi}' + 2\mathcal{H}\boldsymbol{\phi}' + a^2 \mathbf{G}^{-1}\boldsymbol{\nabla}^T V = 0, \tag{42}$$

the right-hand side of equation (40) for Φ can be rewritten as

$$\begin{aligned} -\kappa^2 a^2 (\boldsymbol{\nabla} V\, \boldsymbol{\delta\phi}) &= -\kappa^2 a^2 (\mathbf{G}^{-1}\boldsymbol{\nabla}^T V)\cdot \boldsymbol{\delta\phi} = \kappa^2 \left(\mathcal{D}_\eta \boldsymbol{\phi}' + 2\mathcal{H}\boldsymbol{\phi}'\right)\cdot \Big(\mathbf{e}_1(\mathbf{e}_1\cdot\boldsymbol{\delta\phi}) + \boldsymbol{\delta\phi}^{\perp}\Big) \\ &= 2(\Phi' + \mathcal{H}\Phi)\left(\frac{1}{|\boldsymbol{\phi}'|}(\mathcal{D}_\eta\boldsymbol{\phi}')\cdot\mathbf{e}_1 + 2\mathcal{H}\right) + \kappa^2 (\mathcal{D}_\eta \boldsymbol{\phi}')\cdot\boldsymbol{\delta\phi}^{\perp}, \end{aligned} \tag{43}$$

where we used the definition of the projection operators (35). Inserting this expression in (40) and realizing that $|\boldsymbol{\phi}'|'\,|\boldsymbol{\phi}'| = (\mathcal{D}_\eta\boldsymbol{\phi}')\cdot\boldsymbol{\phi}'$, we get

$$\Phi'' + 2\left(\mathcal{H} - \frac{|\boldsymbol{\phi}'|'}{|\boldsymbol{\phi}'|}\right)\Phi' + 2\left(\mathcal{H}' - \mathcal{H}\frac{|\boldsymbol{\phi}'|'}{|\boldsymbol{\phi}'|}\right)\Phi - \Delta\Phi = \kappa^2 (\mathcal{D}_\eta\boldsymbol{\phi}')\cdot\boldsymbol{\delta\phi}^{\perp}. \tag{44}$$

Notice that, apart from the right-hand side, equation (44) looks identical to equation (17) for the single field case. However, it is exactly this right-hand side which makes it necessary to solve a coupled system of differential equations, as opposed to the decoupled system in the single field case. In this case we need the equation of motion for the scalar field fluctuations,

$$\left(\mathcal{D}_\eta^2 + 2\mathcal{H}\mathcal{D}_\eta - \Delta - \mathbf{R}(\boldsymbol{\phi}', \boldsymbol{\phi}') + a^2\mathbf{M}^2(\boldsymbol{\phi})\right)\boldsymbol{\delta\phi} = 4\Phi'\boldsymbol{\phi}' - 2a^2\Phi\,\mathbf{G}^{-1}\boldsymbol{\nabla}^T V, \tag{45}$$

where we have introduced the mass-matrix

$$\mathbf{M}^2 \equiv \mathbf{G}^{-1}\boldsymbol{\nabla}^T\boldsymbol{\nabla} V, \qquad (\mathbf{M}^2)^a{}_b(\boldsymbol{\phi}) = G^{ac}(\boldsymbol{\phi}) V_{;cb}(\boldsymbol{\phi}). \tag{46}$$

[3] A similar decomposition in the case of two field inflation was simultaneously developed in [6].

Notice that $\mathbf{R}(\boldsymbol{\phi}', \boldsymbol{\phi}') = a^2\mathbf{R}(\dot{\boldsymbol{\phi}}, \dot{\boldsymbol{\phi}})$, so that it is possible to absorb the curvature term into an effective mass matrix. Equation (45) is the multiple field generalization of (20). However, we do not need the total perturbations, but only the perpendicular part, as can be seen from (44). This system of equations for Φ and $\boldsymbol{\delta\phi}^\perp$ we will analyze in section 3.4, but before we do that we consider the background equations during slow-roll inflation.

3.3 Multiple field slow-roll functions

Slow-roll inflation is driven by a flat scalar field potential that acts as an effective cosmological constant because of the small slope. In the case of a single scalar field, the notion of slow roll is well-established (see e.g. [10, 14, 11]). In this paper we generalize this concept to multiple scalar fields in a geometrical way. Afterwards we discuss how our slow-roll functions are related to the well-known single field slow-roll parameters.

To define the slow-roll functions we use comoving time t, since then the background equation does not contain the rapidly changing scale factor a. The background equation of motion (42), the Friedmann equation (3), and equation (8) read in comoving time

$$\mathcal{D}_t\dot{\boldsymbol{\phi}} + 3H\dot{\boldsymbol{\phi}} + \mathbf{G}^{-1}\boldsymbol{\nabla}^T V = 0, \qquad H^2 = \frac{1}{3}\kappa^2\left(\frac{1}{2}|\dot{\boldsymbol{\phi}}|^2 + V\right), \qquad \dot{H} = -\frac{1}{2}\kappa^2|\dot{\boldsymbol{\phi}}|^2. \tag{47}$$

The system is said to be in the slow-roll regime if $|\mathcal{D}_t\dot{\boldsymbol{\phi}}| \ll |3H\dot{\boldsymbol{\phi}}|$ and $\frac{1}{2}|\dot{\boldsymbol{\phi}}|^2 \ll V$. A more precise definition is given below (51). Using (47) the last condition can also be written as $(-\dot{H}) \ll \frac{3}{2}H^2$.

The functions

$$\tilde{\epsilon}(\boldsymbol{\phi}) \equiv -\frac{\dot{H}}{H^2}, \qquad \tilde{\boldsymbol{\eta}}(\boldsymbol{\phi}) \equiv \frac{\boldsymbol{\phi}^{(2)}}{H|\dot{\boldsymbol{\phi}}|}, \qquad \text{and} \qquad \tilde{\boldsymbol{\xi}}(\boldsymbol{\phi}) \equiv \frac{\boldsymbol{\phi}^{(3)}}{H^2|\dot{\boldsymbol{\phi}}|} \tag{48}$$

can be defined whether or not slow roll is valid. Since both $\tilde{\boldsymbol{\eta}}$ and $\tilde{\boldsymbol{\xi}}$ are vectors, they can be decomposed in components using the unit vectors defined in section 3.1. The components of these vectors in the directions $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ are given by

$$\begin{aligned} &\tilde{\eta}^\parallel = \mathbf{e}_1 \cdot \tilde{\boldsymbol{\eta}} = \frac{\mathcal{D}_t\dot{\boldsymbol{\phi}} \cdot \dot{\boldsymbol{\phi}}}{H|\dot{\boldsymbol{\phi}}|^2}, \qquad \tilde{\eta}^\perp = \mathbf{e}_2 \cdot \tilde{\boldsymbol{\eta}} = \frac{|(\mathcal{D}_t\dot{\boldsymbol{\phi}})^\perp|}{H|\dot{\boldsymbol{\phi}}|}, \\ &\tilde{\xi}^\parallel = \mathbf{e}_1 \cdot \tilde{\boldsymbol{\xi}} = \frac{\mathcal{D}_t^2\dot{\boldsymbol{\phi}} \cdot \dot{\boldsymbol{\phi}}}{H^2|\dot{\boldsymbol{\phi}}|^2}, \qquad \tilde{\xi}_2 = \mathbf{e}_2 \cdot \tilde{\boldsymbol{\xi}} = \frac{\mathbf{e}_2 \cdot \mathcal{D}_t^2\dot{\boldsymbol{\phi}}}{H^2|\dot{\boldsymbol{\phi}}|}, \qquad \tilde{\xi}_3 = \mathbf{e}_3 \cdot \tilde{\boldsymbol{\xi}} = \frac{|\mathbf{P}_2^\perp(\mathcal{D}_t^2\dot{\boldsymbol{\phi}})|}{H^2|\dot{\boldsymbol{\phi}}|}. \end{aligned} \tag{49}$$

Since $\tilde{\boldsymbol{\xi}}$ in general has two directions perpendicular to $\mathbf{e}_1$, we cannot use the ambiguous notation $\tilde{\xi}^\perp$. However, since $\tilde{\boldsymbol{\xi}}$ is a vector, $\tilde{\boldsymbol{\xi}}^\perp$ is defined. The components of $\tilde{\boldsymbol{\xi}}$ are not needed for the background equations, but will appear in the equations for the perturbations, see section 3.4. The functions $\tilde{\epsilon}$, $\tilde{\eta}^\perp$, and $\tilde{\xi}_3$ are non-negative, while $\tilde{\eta}^\parallel$, $\tilde{\xi}^\parallel$, and $\tilde{\xi}_2$ can also be negative (apart from the sign, $\tilde{\eta}^\parallel$ is equal to $\frac{|(\mathcal{D}_t\dot{\boldsymbol{\phi}})^\parallel|}{H|\dot{\boldsymbol{\phi}}|}$).

In terms of these functions the Friedmann equation (47) reads

$$H = \frac{\kappa}{\sqrt{3}}\sqrt{V}\left(1 - \frac{1}{3}\tilde{\epsilon}\right)^{-1/2}. \tag{50}$$

For a positive potential V the function $\tilde{\epsilon} < 3$, as can be seen from its definition. Inserting this expression and the functions (48) into the background equation gives

$$\dot{\boldsymbol{\phi}} + \frac{2}{\sqrt{3}}\frac{1}{\kappa}\mathbf{G}^{-1}\boldsymbol{\nabla}^T\sqrt{V} = -\sqrt{\frac{2}{3}}\sqrt{V}\,\frac{\sqrt{\tilde{\epsilon}}}{1-\frac{1}{3}\tilde{\epsilon}}\left(\frac{1}{3}\tilde{\boldsymbol{\eta}} + \frac{\frac{1}{3}\tilde{\epsilon}\,\mathbf{e}_1}{1+\sqrt{1-\frac{1}{3}\tilde{\epsilon}}}\right). \tag{51}$$

From this equation and the one above, both of which are still exact, we can define precisely what is meant by slow roll. Slow roll is valid if $\tilde{\epsilon}$, $\sqrt{\tilde{\epsilon}}\,\tilde{\eta}^{\|}$ and $\sqrt{\tilde{\epsilon}}\,\tilde{\eta}^{\perp}$ are (much) smaller than unity. For this reason $\tilde{\epsilon}$, $\tilde{\eta}^{\|}$ and $\tilde{\eta}^{\perp}$ are called slow-roll functions. However, $\tilde{\eta}^{\|}$ and $\tilde{\eta}^{\perp}$ could even be somewhat larger than one during slow roll, if $\tilde{\epsilon}$ is sufficiently small. On the other hand, we will often use the somewhat stronger condition that $\tilde{\epsilon}$, $\tilde{\eta}^{\|}$ and $\tilde{\eta}^{\perp}$ have to be small individually. The components of $\tilde{\boldsymbol{\xi}}$ are called second order slow-roll functions, and they are assumed to be of an order comparable to $\tilde{\epsilon}^2$, $\tilde{\epsilon}\tilde{\eta}^{\|}$, etc. If slow roll is valid, we can expand in powers of these slow-roll functions. For example, we can expand the previous equation to lowest non-zero order in slow roll, which gives

$$\dot{\boldsymbol{\phi}} + \frac{2}{\sqrt{3}}\frac{1}{\kappa}\mathbf{G}^{-1}\boldsymbol{\nabla}^T\sqrt{V} = -\sqrt{\frac{2}{27}}\sqrt{V}\,\sqrt{\tilde{\epsilon}}\left[\frac{1}{2}\tilde{\epsilon}\,\mathbf{e}_1 + \tilde{\boldsymbol{\eta}} + \mathcal{O}(\tilde{\eta}^{\|}\tilde{\epsilon}, \tilde{\eta}^{\perp}\tilde{\epsilon}, \tilde{\epsilon}^2)\right]. \tag{52}$$

Next we derive some useful expressions for the slow-roll functions in terms of conformal time. Using (49) and (35) we find

$$\begin{aligned}
&\frac{(\mathcal{D}_\eta\boldsymbol{\phi}')^{\perp}}{|\boldsymbol{\phi}'|} = \mathcal{H}\tilde{\boldsymbol{\eta}}^{\perp}, && && \frac{(\mathcal{D}_\eta^2\boldsymbol{\phi}')^{\perp}}{|\boldsymbol{\phi}'|} = \mathcal{H}^2(3\tilde{\boldsymbol{\eta}} + \tilde{\boldsymbol{\xi}})^{\perp}, \\
&\mathcal{H}' = \mathcal{H}^2(1-\tilde{\epsilon}), && \frac{|\boldsymbol{\phi}'|'}{|\boldsymbol{\phi}'|} = \mathcal{H}(1+\tilde{\eta}^{\|}), && \frac{|\boldsymbol{\phi}'|''}{|\boldsymbol{\phi}'|} = \mathcal{H}^2\left(2-\tilde{\epsilon}+3\tilde{\eta}^{\|}+(\tilde{\eta}^{\perp})^2+\tilde{\xi}^{\|}\right).
\end{aligned} \tag{53}$$

Here we differentiated the expression $|\boldsymbol{\phi}'|'|\boldsymbol{\phi}'| = \mathcal{D}_\eta\boldsymbol{\phi}'\cdot\boldsymbol{\phi}'$ with respect to η to find an expression for $|\boldsymbol{\phi}'|''$. Differentiating the slow-roll functions with respect to conformal time η and using some of the above results we find

$$\tilde{\epsilon}' = 2\mathcal{H}\tilde{\epsilon}(\tilde{\epsilon}+\tilde{\eta}^{\|}), \quad (\tilde{\eta}^{\|})' = \mathcal{H}[\tilde{\xi}^{\|} + (\tilde{\eta}^{\perp})^2 + \tilde{\epsilon}\tilde{\eta}^{\|} - (\tilde{\eta}^{\|})^2], \quad \mathcal{D}_\eta\tilde{\boldsymbol{\eta}} = \mathcal{H}[\tilde{\boldsymbol{\xi}} + (\tilde{\epsilon}-\tilde{\eta}^{\|})\tilde{\boldsymbol{\eta}}]. \tag{54}$$

The slow-roll functions (48) are all defined as functions of covariant derivatives of the velocity $\dot{\boldsymbol{\phi}}$, the velocity $\dot{\boldsymbol{\phi}}$ itself, and the Hubble parameter H. If the zeroth order slow-roll approximation works well, that is if the right-hand side of (51) can be neglected, as well as the $\tilde{\epsilon}$ in (50), then we can use these two equations to eliminate $\dot{\boldsymbol{\phi}}$ and H in favour of the potential V. This is the way the conventional single field slow-roll parameters are defined. However, this conventional definition has the disadvantage that the slow-roll conditions become consistency checks. Hence, while we can expand the exact equations in powers of the slow-roll functions, that is impossible by construction with the conventional slow-roll parameters.[4] Therefore, we only show what our slow-roll functions look like in terms of the

[4] In the context of single field inflation this fact was noted before and discussed in detail in [11].

potential in this approximation, but we do not adopt it as the definition:

$$\tilde{\epsilon} = \frac{1}{2\kappa^2}\frac{|\boldsymbol{\nabla} V|^2}{V^2}, \qquad \tilde{\eta}^{\|} - \tilde{\epsilon} = -\frac{1}{\kappa^2}\frac{\boldsymbol{\nabla} V\,\mathbf{M}^2\mathbf{G}^{-1}\boldsymbol{\nabla}^T V}{V|\boldsymbol{\nabla} V|^2} = -\frac{1}{\kappa^2}\frac{\mathrm{tr}\left[\left(\mathbf{M}^2\right)^{\|\|}\right]}{V},$$

$$\tilde{\eta}^{\perp} = \frac{1}{\kappa^2}\frac{|\mathbf{P}^{\perp}\mathbf{M}^2\mathbf{G}^{-1}\boldsymbol{\nabla}^T V|}{V\,|\boldsymbol{\nabla} V|} = \frac{1}{\kappa^2}\frac{\sqrt{\mathrm{tr}\left[(\mathbf{M}^2)^{\|\perp}\,(\mathbf{M}^2)^{\perp\|}\right]}}{V}. \tag{55}$$

The effective mass matrix $\mathbf{M}^2$ is defined in (46). Here $\mathbf{P}^{\|}$ projects along the direction determined by $\boldsymbol{\nabla} V$, which is to lowest order identical to the direction of $\boldsymbol{\phi}'$.

In order to avoid confusion and for later comparison, we finish this section by very explicitly comparing the slow-roll functions we defined in (48) with the ones conventionally used in the single field case, ϵ and η:

$$\epsilon = \frac{1}{2\kappa^2}\frac{V_{,\phi}^2}{V^2} = \tilde{\epsilon}, \qquad \eta = \frac{1}{\kappa^2}\frac{V_{,\phi\phi}}{V} = -\tilde{\eta}^{\|} + \tilde{\epsilon}, \tag{56}$$

where the last equalities in both equations are only valid to lowest order in the slow-roll approximation. Of course, $\tilde{\eta}^{\perp}$ does not exist in the single field case, as there are no other directions than the parallel one.

3.4 Decoupling of the perturbation equations

In this section we analyze the perturbation equations. Starting point are equations (44) for Φ and (45) for $\boldsymbol{\delta\phi}$. First we rewrite these equations in such a way that we can draw some important conclusions using the slow-roll functions defined in the previous section. It turns out that the equation for the (redefined) gravitational potential decouples from the field perturbations up to first order in slow roll. Next we concentrate on solving this equation.

We can write the system of the perturbation equations as a homogeneous matrix equation for the vector $(\Phi, \boldsymbol{\delta\phi}^{\perp})$. However, to remove the first order derivative term from the equation of motion for Φ we define new variables:

$$u \equiv \frac{a}{\kappa^2|\boldsymbol{\phi}'|}\,\kappa^{-1}\Phi = \frac{1}{\sqrt{2}}\frac{a}{\kappa\mathcal{H}\sqrt{\tilde{\epsilon}}}\,\kappa^{-1}\Phi, \qquad \boldsymbol{\delta}\mathbf{v} \equiv \frac{a}{\kappa^2|\boldsymbol{\phi}'|}\,\boldsymbol{\delta\phi}^{\perp} = \frac{1}{\sqrt{2}}\frac{a}{\kappa\mathcal{H}\sqrt{\tilde{\epsilon}}}\,\boldsymbol{\delta\phi}^{\perp}. \tag{57}$$

The additional factor of κ^{-1} in the definition of u has been included to make the mass dimension of both u and $\boldsymbol{\delta}\mathbf{v}$ equal to one. Another important point to note is that we chose our redefinitions in such a way that no relative slow-roll factors have been introduced in the relation between u and $\boldsymbol{\delta}\mathbf{v}$ as compared to the relation between Φ and $\boldsymbol{\delta\phi}^{\perp}$.

We derive the u component of the matrix equation here, as it is the component that we use most. The equation for $\boldsymbol{\delta}\mathbf{v}$ is more complicated, which is why we only give the result here, and refer the reader to the appendix for the derivation. We start with rewriting equation (44) in terms of u and obtain

$$u'' - \Delta u + \left(\frac{|\boldsymbol{\phi}'|''}{|\boldsymbol{\phi}'|} - 2\left(\frac{|\boldsymbol{\phi}'|'}{|\boldsymbol{\phi}'|}\right)^2 - \mathcal{H}^2 + \mathcal{H}'\right) u = \kappa|\boldsymbol{\phi}'|\,\frac{(\mathcal{D}_\eta\boldsymbol{\phi}')^{\perp}}{|\boldsymbol{\phi}'|}\cdot\frac{a}{\kappa^2|\boldsymbol{\phi}'|}\boldsymbol{\delta\phi}^{\perp}. \tag{58}$$

Using the conformal time version of the third equation in (47),

$$\mathcal{H}^2 - \mathcal{H}' = \frac{1}{2}\kappa^2|\boldsymbol{\phi}'|^2, \tag{59}$$

and the definitions of $\boldsymbol{\delta}\mathbf{v}$ and the slow-roll functions (53) we can rewrite this as

$$u'' - \Delta u + \mathcal{H}^2 \left(-2\tilde{\epsilon} - \tilde{\eta}^{\|} - 2(\tilde{\eta}^{\|})^2 + (\tilde{\eta}^{\perp})^2 + \tilde{\xi}^{\|}\right) u = \sqrt{2}\,\mathcal{H}^2 \tilde{\epsilon}^{\frac{1}{2}} \tilde{\boldsymbol{\eta}}^{\perp} \cdot \boldsymbol{\delta}\mathbf{v}. \tag{60}$$

Combining this equation for u with the result for $\boldsymbol{\delta}\mathbf{v}$ from the appendix (184) we get the following, still exact, matrix equation for the perturbations:

$$(\mathfrak{D}_{\mathbf{0}} + \mathfrak{D}_{\mathbf{1}} + \mathfrak{D}_{\mathbf{2}}) \begin{pmatrix} u \\ \boldsymbol{\delta}\mathbf{v} \end{pmatrix} = \mathbf{0}. \tag{61}$$

Here $\mathfrak{D}_{\mathbf{0}}$, $\mathfrak{D}_{\mathbf{1}}$, and $\mathfrak{D}_{\mathbf{2}}$ contain only terms up to order zero, one, and two in the slow-roll functions, respectively. With our choice of the slow-roll functions no higher orders occur. They are given by:

$$\mathfrak{D}_{\mathbf{0}} \equiv \begin{pmatrix} \partial_\eta^2 - \Delta & 0 \\ 0 & \mathcal{D}_\eta^2 + 2\mathcal{H}\mathcal{D}_\eta - \Delta - \mathbf{R}(\boldsymbol{\phi}', \boldsymbol{\phi}') + a^2 (\mathbf{M}^2)^{\perp\perp} \end{pmatrix}, \tag{62}$$

$$\mathfrak{D}_{\mathbf{1}} \equiv \begin{pmatrix} -\mathcal{H}^2 (2\tilde{\epsilon} + \tilde{\eta}^{\|}) & 0 \\ 2\sqrt{2}\,\tilde{\epsilon}^{-1/2} \tilde{\boldsymbol{\eta}}^{\perp} \Delta & \left(\tilde{\eta}^{\|} \mathbb{1} + \mathbf{e}_1 (\tilde{\boldsymbol{\eta}}^{\perp})^{\dagger}\right) \left(2\mathcal{H}\mathcal{D}_\eta + 3\mathcal{H}^2\right) \end{pmatrix}, \tag{63}$$

$$\mathfrak{D}_{\mathbf{2}} \equiv \mathcal{H}^2 \begin{pmatrix} -2(\tilde{\eta}^{\|})^2 + (\tilde{\eta}^{\perp})^2 + \tilde{\xi}^{\|} & -\sqrt{2}\,\tilde{\epsilon}^{1/2} (\tilde{\boldsymbol{\eta}}^{\perp})^{\dagger} \\ 0 & \left((\tilde{\eta}^{\perp})^2 + \tilde{\xi}^{\|}\right) \mathbb{1} + 4\tilde{\boldsymbol{\eta}}^{\perp} (\tilde{\boldsymbol{\eta}}^{\perp})^{\dagger} + \mathbf{e}_1 (\tilde{\boldsymbol{\xi}}^{\perp})^{\dagger} \end{pmatrix}. \tag{64}$$

From these equations one can draw the important conclusion that the redefined gravitational potential u decouples from the perpendicular components of the field $\boldsymbol{\delta}\mathbf{v}$ up to and including first order in slow roll. The resulting equation for u may be written in Fourier components (7) as

$$u''_{\mathbf{k}} + \left(k^2 - \frac{\theta''}{\theta}\right) u_{\mathbf{k}} = 0, \qquad \text{with} \qquad \theta \equiv \frac{\mathcal{H}}{a|\boldsymbol{\phi}'|} = \frac{\kappa}{\sqrt{2}} \frac{1}{a\sqrt{\tilde{\epsilon}}}. \tag{65}$$

From the second expression for θ and (54) it follows that

$$\frac{\theta'}{\theta} = -\mathcal{H}\left(1 + \tilde{\epsilon} + \tilde{\eta}^{\|}\right), \qquad \frac{\theta''}{\theta} = \mathcal{H}^2 \left(2\tilde{\epsilon} + \tilde{\eta}^{\|} + 2(\tilde{\eta}^{\|})^2 - (\tilde{\eta}^{\perp})^2 - \tilde{\xi}^{\|}\right). \tag{66}$$

We see that we got back equation (19), which was derived in [19]. However, here we have proved that this equation is not only valid in the single field case, but also in the multiple field case, up to and including first order in slow roll, i.e. neglecting the term that is proportional to $\sqrt{\tilde{\epsilon}}\,\tilde{\eta}^{\perp}$. This is precisely one of the combinations of slow-roll functions that is small if slow roll is valid. Notice that even if slow roll is no longer valid, the specific combination $\sqrt{\tilde{\epsilon}}\,\tilde{\eta}^{\perp}$ may still be small. This means that slow-roll inflation ends because one of the other combinations of slow-roll functions becomes large. We treat an example of this later on in section 5.1.2: a quadratic potential where the mass difference between the lightest mass and the other masses is large enough. On the other hand, we see that the equation for the field perturbations $\boldsymbol{\delta}\mathbf{v}$ already depends on the solution for u at lower order, although compared with other terms this coupling term will become smaller during inflation because it does not contain $\mathcal{H}^2$. We defined u and $\boldsymbol{\delta}\mathbf{v}$ in terms of Φ and $\boldsymbol{\delta}\boldsymbol{\phi}^{\perp}$ in equation (57) with the

same powers of $\tilde{\epsilon}$, so that the decoupling holds to the same order for Φ and $\boldsymbol{\delta\phi}^{\perp}$. In the literature, e.g. [24, 20, 6], this leading order perturbation in u is called the adiabatic mode, while the perturbations associated with the perpendicular field components are related to the so-called isocurvature modes. In this paper we restrict ourselves to the adiabatic mode, but in a subsequent paper we plan to investigate the perpendicular field equations and the isocurvature modes.

The decoupled equation for u cannot be solved analytically in general. However, we can solve equation (65) for u exactly in the two limits of large and small k, as was observed in [19]. The two linearly independent solutions are denoted by $e_{\mathbf{k}}$ and its complex conjugate $e_{\mathbf{k}}^*$. In the small wavelength limit the solution is

$$e_{\mathbf{k}}(\eta) = e^{ik(\eta-\eta_k)} \qquad \text{for} \qquad k^2 \gg \left|\frac{\theta''}{\theta}\right|, \tag{67}$$

where the time η_k is defined as the time when k^2 is equal to $|\theta_k''/\theta_k|$. (We use the non-bold subscript $_k$ to indicate that a quantity is evaluated at $\eta = \eta_k$.) The normalization of the two independent solutions is such that their Wronskian satisfies

$$W(e_{\mathbf{k}}^*(\eta), e_{\mathbf{k}}(\eta)) = e_{\mathbf{k}}^*(\eta)e_{\mathbf{k}}'(\eta) - e_{\mathbf{k}}^{*\prime}(\eta)e_{\mathbf{k}}(\eta) = i\,2k. \tag{68}$$

In the limit of large wavelengths the solution of (65) is

$$e_{\mathbf{k}}(\eta) = C_{\mathbf{k}}\theta(\eta) + D_{\mathbf{k}}\theta(\eta)\int_{\eta_k}^{\eta}\frac{\mathrm{d}\eta'}{\theta^2(\eta')} \qquad \text{for} \qquad k^2 \ll \left|\frac{\theta''}{\theta}\right|, \tag{69}$$

which can be rewritten in terms of comoving time as

$$e_{\mathbf{k}}(t) = \frac{H}{a|\dot{\boldsymbol{\phi}}|}\left(C_{\mathbf{k}} - D_{\mathbf{k}}\frac{2a_k}{\kappa^2 H_k}\right) + \frac{2}{\kappa^2|\dot{\boldsymbol{\phi}}|}\,D_{\mathbf{k}}\left(1 - \frac{H}{a}\int_{t_k}^{t} a(\tau)\,\mathrm{d}\tau\right). \tag{70}$$

Here we have used the definition of θ, the relation $\tilde{\epsilon} = (1/H)^{\cdot}$, and integration by parts.

Simply joining the two limits in a continuously differentiable way at $\eta = \eta_k$ determines the integration constants:

$$C_{\mathbf{k}} = \frac{1}{\theta_k} \qquad \text{and} \qquad D_{\mathbf{k}} = \theta_k\left[ik - \frac{\theta_k'}{\theta_k}\right]. \tag{71}$$

This joint solution gives a good approximation of the true solution of (65) if slow roll is valid, because then the transition region is small, as we now show. First, we define the transition region as that region where the terms k^2 and θ''/θ in the equation of motion for u are of the same order, i.e. within a factor α^2 of each other with $\alpha^2 \sim 10$: $\alpha^{-2}k^2 < |\theta''/\theta| < \alpha^2 k^2$. We want to know what the size of this transition region is in terms of η, compared to the characteristic time scale $2\pi/k$ of the solution (67). To lowest order in slow roll $|\theta''/\theta| = a^2H_k^2(2\tilde{\epsilon}_k + \tilde{\eta}_k^{\|})$ and $k^2 = |\theta_k''/\theta_k| = a_k^2H_k^2(2\tilde{\epsilon}_k + \tilde{\eta}_k^{\|})$. Hence if we define η_- as the beginning and η_+ as the end time of the transition region, we have

$$\alpha^{\pm 1} \equiv \left|\frac{(\theta''/\theta)_{\eta_\pm}}{(\theta''/\theta)_{\eta_k}}\right|^{1/2} = \frac{a(\eta_\pm)}{a_k}. \tag{72}$$

By integrating the approximation $\mathcal{H}(\eta) = a'(\eta)/a(\eta) \approx a(\eta)H_k$ around the transition time η_k, we find that $1 - a_k/a(\eta) = a_k H_k(\eta - \eta_k)$. This leads to the following leading order expression for the ratio of the duration of the transition and the period of the oscillation:

$$\frac{\eta_+ - \eta_-}{2\pi/k} = \frac{k}{a_k H_k}\frac{\alpha - \alpha^{-1}}{2\pi} = \sqrt{2\tilde{\epsilon}_k + \tilde{\eta}_k^{\|}}\,\frac{\alpha - \alpha^{-1}}{2\pi}. \tag{73}$$

Hence if the slow-roll functions are all individually smaller than 0.01, which is a reasonable assumption as we will show in the examples in section 5, then the transition region does not last longer than approximately a tenth of an oscillation period.

We can also discuss the accuracy of our treatment of the transition between the small and large wavelength limits in a different way. As we discuss in the next section, the important quantity is $|D_\mathbf{k}|^2$:

$$|D_\mathbf{k}|^2 = \theta_k^2\left(\left(\frac{\theta_k'}{\theta_k}\right)^2 + k^2\right) = \theta_k^2\left(\left(\frac{\theta_k'}{\theta_k}\right)^2 + \left|\frac{\theta_k''}{\theta_k}\right|\right) = \frac{\kappa^2}{2}\frac{H_k^2}{\tilde{\epsilon}_k}, \tag{74}$$

where in the last step we have included only the leading order term in slow roll. We estimate the maximum error in $|D_\mathbf{k}|^2$ as the difference between the values of $|D_\mathbf{k}|^2$ determined by matching at η_- and at η_+:

$$\frac{|D_\mathbf{k}|^2_- - |D_\mathbf{k}|^2_+}{|D_\mathbf{k}|^2_k} \approx \frac{\left(|D_\mathbf{k}|^2\right)'_k(\eta_- - \eta_+)}{|D_\mathbf{k}|^2_k} = 2\left(2\tilde{\epsilon}_k + \tilde{\eta}_k^{\|}\right)\left(\alpha - \alpha^{-1}\right) \tag{75}$$

to leading order in slow roll, where we made use of (54) and (73). From this we conclude that we can indeed only give the leading order term for $|D_\mathbf{k}|^2$ in (74), since corrections at the next order are expected for a more accurate treatment of the transition region. For the single field case an expression for $|D_\mathbf{k}|^2$ that is accurate up to and including next-to-leading order terms was obtained in [27, 17].

We finish this section with an argument why the difference between η_k and η_H does not matter in the expression for $|D_\mathbf{k}|^2$ to leading order. Here η_H is the time of horizon crossing, defined by $k^2 = \mathcal{H}^2$, which is conventionally used in the literature. On the other hand, η_k is the time when the solution of the differential equation (65) changes its behaviour, defined by $k^2 = |\theta''/\theta| = \mathcal{H}^2(2\tilde{\epsilon} + \tilde{\eta}^{\|} + \dots)$, which we use to compute $|D_\mathbf{k}|^2$. The difference between these two expressions for k^2 can be quite large during slow roll. However, just as in the previous paragraph, the relevant quantity is

$$\frac{|D_\mathbf{k}|^2_H - |D_\mathbf{k}|^2_k}{|D_\mathbf{k}|^2_k} \approx \frac{\left(|D_\mathbf{k}|^2\right)'_k(\eta_H - \eta_k)}{|D_\mathbf{k}|^2_k} = 2\sqrt{2\tilde{\epsilon}_k + \tilde{\eta}_k^{\|}} \tag{76}$$

to lowest order in slow roll. Here we used that $a_H = a_k\sqrt{2\tilde{\epsilon}_k + \tilde{\eta}_k^{\|}}$, as follows from the definitions above, and inserted this into the expression in the text above (73) to calculate $\eta_H - \eta_k$. We see that corrections to $|D_\mathbf{k}|^2$ because of the difference between η_k and η_H are of higher order in slow roll, if we take the slow-roll functions to be small individually.

4 Quantum correlation function of the Newtonian potential

The quantum correlation function $\langle \Phi(\mathbf{x},\eta)\Phi(\mathbf{x}+\mathbf{r},\eta)\rangle$ during inflation with multiple scalar fields is the central object of study in this section. In particular, we obtain an expression for the correlation function at the end of inflation. Before the actual calculation can be performed various questions have to be addressed. First of all, can one simply quantize the Newtonian potential? Another question is which quantum state is appropriate for the computation of the correlator.

In equation (57) of section 3.4 we have introduced the variable u and shown that it decouples from the perpendicular scalar field fluctations $\boldsymbol{\delta}\mathbf{v}$ if corrections of the order of $\sqrt{\tilde{\epsilon}}\,\tilde{\eta}^{\perp}$ are neglected, and that its equation of motion is then given by (65). However, the Newtonian potential Φ does not correspond to a physical degree of freedom within the metric; only the graviton states represent physical degrees of freedom and can be quantized in an on-shell quantization procedure. (Alternatively, one can use BRS quantization to avoid making explicit gauge choices [1, 8].) The scalar field perturbations $\boldsymbol{\delta\phi}$ on the other hand are physical degrees of freedom, hence they should be quantized. But we are not interested in all these scalar perturbations: only those that are directly related to the Newtonian potential to this order in slow roll. The relevant multiple field generalization of the variable v introduced in [19] is

$$v \equiv \frac{a}{\kappa}\left(\mathbf{e}_1 \cdot \boldsymbol{\delta\phi} + \frac{|\boldsymbol{\phi}'|}{\mathcal{H}}\Phi\right) = 2\left(u' - \frac{\theta'}{\theta}u\right), \tag{77}$$

where we have used (41) together with the definitions (57) and (65). A slightly different form of this variable (without the factor a/κ) is sometimes refered to as Mukhanov-Sasaki variable [18, 25]. The equation of motion for v can be found using the exact equation of motion for u, i.e. (60) combined with (66), and the expressions for the derivatives of $\mathcal{H}$ (53) and the slow-roll functions (54):

$$v'' - \Delta v - \frac{(1/\theta)''}{1/\theta}v = 2\sqrt{2}\,\mathcal{H}^2\sqrt{\tilde{\epsilon}}\left[\mathcal{H}\left((3+\tilde{\epsilon}+\tilde{\eta}^{\|})\tilde{\boldsymbol{\eta}} + \tilde{\boldsymbol{\xi}}\right)^{\perp} \cdot \boldsymbol{\delta}\mathbf{v} + \tilde{\boldsymbol{\eta}} \cdot \mathcal{D}_\eta \boldsymbol{\delta}\mathbf{v}\right]. \tag{78}$$

The definition of v also includes a term with Φ, which ensures that the equation of motion for v can be written in terms of v and $\boldsymbol{\delta}\mathbf{v}$ only. (If we were not in the situation where $\Psi = \Phi$, the definition of v would contain Ψ instead of Φ. This definition is automatically gauge invariant, as can be seen from (2), and therefore it is guaranteed that no non-physical degrees of freedom are quantized.) The scale factor a is introduced to remove the first derivative term in the equation of motion. The result is that at the beginning of inflation, when $k^2 \gg |(1/\theta)''/(1/\theta)|$ as we shall discuss below, the left-hand side of the equation for v is simply the equation of the harmonic oscillator. On the right-hand side we see that v decouples from $\boldsymbol{\delta}\mathbf{v}$ up to the same order in slow roll as u does, provided that $\tilde{\boldsymbol{\xi}}^{\perp}$ is small as well. Hence we know how to quantize v. Moreover, in this limit of large k the equation of motion for v is equal to the one for u (65), up to terms of order $\sqrt{\tilde{\epsilon}}\,\tilde{\boldsymbol{\eta}}$ in slow roll. This means that the quantum operator $\hat{u}$ can be expanded in terms of the same creation and annihilation operators as $\hat{v}$, so that once we have determined the normalization by quantizing v, we can simply return to the equation for u to determine the time evolution and compute the correlator at later times.

In the approximation where v decouples from $\boldsymbol{\delta}\mathbf{v}$, the action for v given by

$$S = \int \mathrm{d}^3\mathbf{x}\mathrm{d}\eta\, \kappa^2 \left[\frac{1}{2}(v')^2 + \frac{1}{2}v\Big(\Delta + \frac{(1/\theta)''}{1/\theta}\Big)v\right] \tag{79}$$

gives rise to the equation of motion (78). We continue by canonically quantizing the variable v at the beginning of inflation $\eta = \eta_i$. In the quantum theory $\hat{v}$ and its canonical momentum $\hat{\pi} = \kappa^2 \hat{v}'$ satisfy the commutation relation

$$[\hat{v}(\mathbf{x},\eta), \hat{\pi}(\mathbf{y},\eta)] = i\delta(\mathbf{x}-\mathbf{y}). \tag{80}$$

At the beginning of inflation the field $\hat{v}$ can be expanded as follows:

$$\hat{v}(\mathbf{x},\eta) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}\sqrt{2k}\,\kappa} \left\{ v_{\mathbf{k}}(\eta) e^{-i\mathbf{k}\mathbf{x}}\, \hat{b}^\dagger_{\mathbf{k}} + v^*_{\mathbf{k}}(\eta) e^{i\mathbf{k}\mathbf{x}}\, \hat{b}_{\mathbf{k}} \right\}, \tag{81}$$

where the time independent creation and annihilation operators $\hat{b}^\dagger_{\mathbf{k}}$ and $\hat{b}_{\mathbf{k}}$ obey the commutation relation

$$[\hat{b}_{\mathbf{k}}, \hat{b}^\dagger_{\mathbf{l}}] = \delta(\mathbf{k}-\mathbf{l}). \tag{82}$$

In order for this expansion to be valid the dimensionless mode functions $v_{\mathbf{k}}(\eta)$ and $v^*_{\mathbf{k}}(\eta)$ have to satisfy the Wronskian condition

$$W(v^*_{\mathbf{k}}(\eta), v_{\mathbf{k}}(\eta)) = v^*_{\mathbf{k}}(\eta)v'_{\mathbf{k}}(\eta) - v^{*\prime}_{\mathbf{k}}(\eta)v_{\mathbf{k}}(\eta) = i\,2k, \tag{83}$$

where the $v_{\mathbf{k}}$ only depend on the length k of $\mathbf{k}$. It can be checked that the equation of motion implies that $\frac{\mathrm{d}}{\mathrm{d}\eta}W(v^*_{\mathbf{k}}, v_{\mathbf{k}}) = 0$, so that the quantization procedure is consistent with the equation of motion; this is guaranteed by using canonical quantization. At the initial stages of inflation the scales k^2 that are observable in the CMBR today are much, much larger than the horizon: $k^2 \gg \mathcal{H}^2$. We assume that this implies that k^2 is also very large compared to the other relevant quantities $|\theta''/\theta|$, $(\theta'/\theta)^2$ and $|(1/\theta)''/(1/\theta)|$, since they are all proportional to $\mathcal{H}^2$, up to factors of order unity or slow roll. Therefore, when inflation starts the variable v can be expanded in terms of the independent solutions $e_{\mathbf{k}}(\eta)$ and $e^*_{\mathbf{k}}(\eta)$ defined in (67). As explained below (78), we can expand $\hat{u}$ in the same way. Introducing the notation

$$\Theta \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \hat{B}_{\mathbf{k}} = \begin{pmatrix} \hat{b}^\dagger_{\mathbf{k}} \\ \hat{b}_{\mathbf{k}} \end{pmatrix}, \qquad E_{\mathbf{k}} = \begin{pmatrix} e_{\mathbf{k}} & e^*_{\mathbf{k}} \end{pmatrix} \qquad \Rightarrow \qquad E'_{\mathbf{k}} = ikE_{\mathbf{k}}\Theta, \tag{84}$$

we may write $\hat{v}$ and $\hat{u}$ as

$$\hat{v}(\mathbf{x},\eta) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}\sqrt{2k}\,\kappa} E_{\mathbf{k}} R_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x}\,\Theta} \hat{B}_{\mathbf{k}}, \qquad \hat{u}(\mathbf{x},\eta) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}} E_{\mathbf{k}} U_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x}\,\Theta} \hat{B}_{\mathbf{k}}. \tag{85}$$

Combining the Wronskian condition for $e_{\mathbf{k}}$ (68), written as $W(E_{\mathbf{k}}) = E_{\mathbf{k}}\Theta E^\dagger_{\mathbf{k}} = i\,2k$, with the Wronskian condition for $v_{\mathbf{k}}$ (83), $W(E_{\mathbf{k}}R_{\mathbf{k}}) = E_{\mathbf{k}}R_{\mathbf{k}}\Theta R^\dagger_{\mathbf{k}}E^\dagger_{\mathbf{k}} = i\,2k$, we see that $R_{\mathbf{k}}$ is an element of the non-compact unitary group $U(1,1)$ defined by

$$R_{\mathbf{k}}\Theta R^\dagger_{\mathbf{k}} = \Theta. \tag{86}$$

In addition the Hermiticity of $\hat{v}$ implies that $R_{\mathbf{k}}$ is an element of $SU(1,1)$. The matrix $R_{\mathbf{k}}$ represents a Bogolubov transformation [4]. Using the relation (77) between v and u, (84) for $E'_{\mathbf{k}}$, and the fact that $k \gg |\theta'/\theta|$, the expansion in quantum mechanical operators leads to

$$U_{\mathbf{k}} = \frac{-i}{(2k)^{3/2}\kappa}\,\Theta R_{\mathbf{k}}. \tag{87}$$

We now derive a general compact expression for the expectation value of the correlator $\langle \hat{u}(\mathbf{x},\eta)\hat{u}^\dagger(\mathbf{x}+\mathbf{r},\eta)\rangle_\rho$ computed in an arbitary state that is represented by the density matrix $\hat{\rho}$. We assume that expectation values $\langle \hat{b}_{\mathbf{k}}\hat{b}_{\mathbf{l}}\rangle_\rho = 0$, etc., if $\mathbf{k} \neq \mathbf{l}$. Then

$$\langle \hat{B}_{\mathbf{k}}\hat{B}^\dagger_{\mathbf{l}}\rangle_\rho = \begin{pmatrix} \langle N_{\mathbf{k}}\rangle_\rho & \langle \hat{b}^2_{\mathbf{k}}\rangle^*_\rho \\ \langle \hat{b}^2_{\mathbf{k}}\rangle_\rho & 1 + \langle \hat{N}_{\mathbf{k}}\rangle_\rho \end{pmatrix} \delta(\mathbf{k}-\mathbf{l}), \tag{88}$$

with the number operator $\hat{N}_{\mathbf{k}} = \hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}$. The two-point correlator becomes

$$\langle \hat{u}(\mathbf{x})\hat{u}^\dagger(\mathbf{x}+\mathbf{r})\rangle_{\hat{\rho}} = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\, E_{\mathbf{k}}\, U_{\mathbf{k}}\, e^{-i\mathbf{k}\mathbf{x}\,\sigma} \begin{pmatrix} \langle \hat{N}_{\mathbf{k}}\rangle_\rho & \langle \hat{b}^2_{\mathbf{k}}\rangle^*_\rho \\ \langle \hat{b}^2_{\mathbf{k}}\rangle_\rho & 1 + \langle \hat{N}_{\mathbf{k}}\rangle_\rho \end{pmatrix} e^{i\mathbf{k}(\mathbf{x}+\mathbf{r})\,\sigma}\, U^\dagger_{\mathbf{k}}\, E^\dagger_{\mathbf{k}}. \tag{89}$$

We illustrate this expression with two examples. The simplest state to consider is a conformal vacuum state density matrix $\hat{\rho}_0 = |0\rangle\langle 0|$ where for all $\mathbf{k}$ the state $|0\rangle$ is annihilated by $\hat{b}_{\mathbf{k}}$. The correlator then takes the form

$$\langle \hat{u}(\mathbf{x})\hat{u}^\dagger(\mathbf{x}+\mathbf{r})\rangle_0 = \int \frac{\mathrm{d}^3\mathbf{k}}{(4\pi\, k)^3\kappa^2}\, \left|(R_{\mathbf{k}})_{22}\, e^*_{\mathbf{k}} - (R_{\mathbf{k}})_{12}\, e_{\mathbf{k}}\right|^2\, e^{-i\mathbf{k}\mathbf{r}}. \tag{90}$$

This shows that $R_{\mathbf{k}}$ represents a Bogolubov transformation. In other words, $R_{\mathbf{k}}$ measures the alignment of the expansion of the field $\hat{u}$ in terms of creation and annihilation operators ($\hat{b}^\dagger_{\mathbf{k}}$ and $\hat{b}_{\mathbf{k}}$) with respect to the Lorentz conformal vacuum. In the Lorentz conformal vacuum all matrices $R_{\mathbf{k}}$ are equivalent to the identity, so that the field expansion takes the familiar Lorentz-invariant form

$$\hat{u}(\mathbf{x},\eta) = \int \frac{\mathrm{d}^3\mathbf{k}}{(4\pi\, k)^{3/2}\kappa} \left\{ e^{i(k\eta - \mathbf{k}\mathbf{x})}\, \hat{b}^\dagger_{\mathbf{k}} + e^{-i(k\eta - \mathbf{k}\mathbf{x})}\, \hat{b}_{\mathbf{k}} \right\}. \tag{91}$$

Here we used the definition of the functions $e_{\mathbf{k}}$ given in (67) for modes with $k^2 \gg |\theta''/\theta|$. The canonical Hamiltonian $\hat{H}$ associated with the action (79) is given by

$$\hat{H} = \int \mathrm{d}^3\mathbf{k}\, \frac{k}{2} \left[\sinh(2\vartheta) \left(e^{i2\delta}\, \hat{b}^\dagger_{\mathbf{k}}\hat{b}^\dagger_{-\mathbf{k}} + \mathrm{h.c.}\right) + \cosh(2\vartheta)\,(2\hat{N}_{\mathbf{k}} + 1)\right], \tag{92}$$

where we wrote

$$R_{\mathbf{k}} = \begin{pmatrix} \cosh\vartheta\, e^{i(\varphi+\delta)} & \sinh\vartheta\, e^{i(\varphi-\delta)} \\ \sinh\vartheta\, e^{-i(\varphi-\delta)} & \cosh\vartheta\, e^{-i(\varphi+\delta)} \end{pmatrix}, \tag{93}$$

which is a representation of the most general form for an element of $SU(1,1)$. Here ϑ, φ and δ are real, and in general depend on $\mathbf{k}$. We see that for $\vartheta = 0$ there is no pair creation of particles and anti-particles (the $\hat{b}^\dagger_{\mathbf{k}}\hat{b}^\dagger_{-\mathbf{k}}$ term drops out), and $\cosh(2\vartheta)$ takes its minimum

value at $\vartheta = 0$. Hence choosing the Lorentz alignment $\vartheta = 0$ seems to lead to a configuration with the usual vacuum properties.

Our second example is a thermal state of temperature $1/\beta$ in Planckian units characterized by

$$\rho_\beta \propto e^{-\beta \hat{H}}, \qquad \hat{H} = \int \mathrm{d}^3\mathbf{k}\, k\, \hat{N}_\mathbf{k}, \qquad \langle \hat{N}_\mathbf{k} \rangle_{\rho_\beta} = \frac{1}{e^{\beta k} - 1}. \tag{94}$$

Here we chose the Lorentz alignment and neglected the (infinite) zero-point energy, which is irrelevant for the definition of the density matrix. In the thermal state we have that $\langle \hat{b}^2_\mathbf{k} \rangle_{\rho_\beta} = 0$, hence we find for the thermal correlator

$$\langle \hat{u}(\mathbf{x}, \eta) \hat{u}(\mathbf{x} + \mathbf{r}, \eta) \rangle_{\rho_\beta} = \int \frac{\mathrm{d}^3\mathbf{k}}{(4\pi\, k)^3 \kappa^2} |e_\mathbf{k}(\eta)|^2 \left(1 + 2\langle \hat{N}_\mathbf{k} \rangle_{\rho_\beta}\right) e^{-i\mathbf{kr}}, \tag{95}$$

where the occupation number $\langle \hat{N}_\mathbf{k} \rangle_{\rho_\beta}$ is given in (94).

Next we argue why taking the vacuum state $|0\rangle$ at the beginning of inflation is a reasonable assumption for the calculation of the density perturbations that we can observe in the CMBR today. Even though perturbations in the CMBR have long wavelengths now, they had very short wavelengths before they went through the horizon during inflation. Therefore, their scale k at the beginning of inflation at t_i is much larger than the Planck scale. It seems a reasonable assumption that modes with momenta very much larger than the Planck scale are not excited at t_i, so that for these modes the vacuum state is a good assumption.[5]

This argument becomes more convincing if we put in some numbers. Assume that inflation starts around the Planck time, $t_i = t_P$, when the horizon is naturally of the order of the Planck scale: $H \approx \kappa^{-1}$. Of course we do not know the exact quantum state at the Planck time, but as a first indication we take a thermal state with a temperature of the order of the Planck energy: $\beta \approx 1$.[6] The conformal Hubble parameter is given by $\mathcal{H} = \kappa a_i e^N H \approx a_i e^N$ with $N = \int_{t_i}^t H \mathrm{d}t$, where in the last step we made the approximation of a constant H during (the beginning of) inflation. Now we evaluate $\mathcal{H}$ at the time when the scale k goes through the horizon, $k = \mathcal{H}$. Suppose that inflation started just before this happens, say $N \approx 2$, and that $a_i \approx 1$ (the universe should at least be a Planck length at the Planck time, and a larger value only makes the argument stronger). Even then we find that the thermal correction is already very small: $2\langle \hat{N}_\mathbf{k} \rangle_{\rho_\beta} \approx 10^{-3}$. In physically more realistic situations the number of e-folds N can easily be of the order of 100, so that this thermal effect is completely negligible because $\exp(\exp 100))$ gives a huge suppression.

As we are making a claim concerning physics at a time when we do not have any solid theoretical or experimental data, we have to make sure that our statement does not depend strongly on the precise choice of the initial time t_i of inflation. However, because $|\theta''/\theta| \ll k^2$ the field equation (65) of u is a plain wave equation, it follows that the Bogolubov transformation between the Fock spaces at different initial times can be neglected. This implies that in our statement there is no fine-tuning problem associated with the starting time of inflation.

[5] There could be a problem with this approach, because our knowledge of physics beyond the Planck scale is extremely poor. In particular, the dispersion relation $\omega(\mathbf{k}) = k$ that we used implicitly might not be valid for large k: there might be a cut-off for large momenta. For a discussion of this trans-Planckian problem and possible cosmological consequences see [15].

[6] Other non-vacuum initial states and possible observable effects have been investigated in [16].

In the remainder of this section we compute the vacuum correlator for the gravitational potential Φ, derive expressions for the amplitude and the slope of the density perturbation spectrum, and compare our results with the literature. Using the Lorentz alignment for the vacuum, the vacuum correlation function for $\Phi = \kappa^3|\dot{\boldsymbol{\phi}}|u$ can be calculated from (90) and (70):

$$\langle\Phi(\mathbf{x},t)\Phi(\mathbf{x}+\mathbf{r},t)\rangle_0 = \frac{1}{2}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi\, k)^3}\,|D_{\mathbf{k}}|^2 \left(1-\frac{H}{a}\int_{t_k}^{t} a(\tau)\,\mathrm{d}\tau\right)^2 e^{-i\mathbf{kr}}, \tag{96}$$

where we have used that for large time $t \gg t_k$ the first term in (70) can be neglected because of the $1/a$ suppression. The only model dependence resides in the norm of the coefficient $|D_{\mathbf{k}}|^2$, given in (74).

Using the fact that $|D_{\mathbf{k}}|^2$ only depends on the length of $\mathbf{k}$, we can perform the integration over the angles and obtain:

$$\langle\Phi(\mathbf{x},t)\Phi(\mathbf{x}+\mathbf{r},t)\rangle_0 = \int_0^\infty \frac{\mathrm{d}k}{k}\frac{\sin kr}{kr}\,|\delta_{\mathbf{k}}(t)|^2, \tag{97}$$

where

$$|\delta_{\mathbf{k}}(t)|^2 = \frac{1}{4\pi^2}|D_{\mathbf{k}}|^2\left(1-\frac{H}{a}\int_{t_k}^{t} a(\tau)\,\mathrm{d}\tau\right)^2. \tag{98}$$

To obtain an estimate of the size of this quantity, we first observe that

$$1-\frac{H}{a}\int_{t_k}^{t} a(\tau)\,\mathrm{d}\tau = \left(\frac{1}{a}\int_{t_k}^{t} a(\tau)\,\mathrm{d}\tau\right)^{\cdot}. \tag{99}$$

Because $H(t)$ and $a(t)$ are positive functions, we see immediately that the left-hand side has to be smaller than or equal to one for $t \geq t_k$. On the other hand, as long as $a(t)$ is positive, and does not grow faster than exponentially, the term within brackets on the right-hand side will be a non-decreasing function of t, so that its time derivative is non-negative. Hence for all cases of interest

$$0 \leq \left(\frac{1}{a}\int_{t_k}^{t} a\,\mathrm{d}\tau\right)^{\cdot} \leq 1 \qquad \Rightarrow \qquad |\delta_{\mathbf{k}}(t)|^2 \leq \frac{1}{4\pi^2}|D_{\mathbf{k}}|^2. \tag{100}$$

As mentioned in the introduction, we extrapolate our result right to the time of recombination for the purpose of comparing it with the results in the literature. In [29] it is shown that this is justified for adiabatic perturbations, which are the only ones we consider here. However, a full discussion of the validity of this extrapolation is beyond the scope of this paper. At the time of recombination during matter domination $a(t) \propto t^{2/3}$, which leads to

$$|\delta_{\mathbf{k}}|^2 = \frac{9}{25}\frac{1}{4\pi^2}|D_{\mathbf{k}}|^2. \tag{101}$$

To leading order in slow roll this is exactly the same result as that obtained in [10], if one takes into account that the δ_H^2 defined in that paper equals $\frac{4}{9}|\delta_{\mathbf{k}}|^2$. The calculation of the slope of the spectrum is analogous to the one presented in [10] and gives to leading order in slow roll:

$$n-1 \equiv \frac{\mathrm{d}\ln|\delta_{\mathbf{k}}|^2}{\mathrm{d}\ln k} = -4\tilde{\epsilon}_k - 2\tilde{\eta}_k^{\parallel} = -6\epsilon_k + 2\eta_k, \tag{102}$$

where in the last step we took the single field limit and switched to the conventional slow-roll parameters (56) for the purpose of comparison. We see that in that case the result is identical to the one in [10]. Comparing with the result in [26] we see, after rewriting it in terms of our slow-roll functions, that all corrections are indeed of higher order.

5 Slow roll with multiple scalar fields

5.1 Slow roll on a flat manifold

In this section we consider some examples of slow-roll inflation with scalar fields living on a flat manifold. Since all flat manifolds are locally isomorphic to a subset of $\mathbb{R}^N$, we assume that the N scalar fields live in the $\mathbb{R}^N$ themselves. In particular, we use the standard basis for $\mathbb{R}^N$. The (zeroth order) slow-roll equation of motion and Friedmann equation for the background quantities are given by

$$\dot{\boldsymbol{\phi}} = -\frac{2}{\sqrt{3}\,\kappa}\,\boldsymbol{\partial}^T\sqrt{V(\boldsymbol{\phi})}, \qquad H = \frac{\kappa}{\sqrt{3}}\sqrt{V(\boldsymbol{\phi})}. \tag{103}$$

We make use of the hat to indicate a unit vector: $\hat{\boldsymbol{\phi}} \equiv \boldsymbol{\phi}/\phi$, with $\phi \equiv \sqrt{\boldsymbol{\phi}^T\boldsymbol{\phi}}$ the length of the vector $\boldsymbol{\phi}$. In the first subsection we consider a quadratic potential where all scalar field components have equal masses, while in the second subsection we focus on the more complicated case of a quadratic potential with an arbitrary mass matrix.

5.1.1 Scalar fields with identical masses on a flat manifold

In this example all masses are assumed to be equal to $\kappa^{-1}m$, so that the mass matrix is proportional to the identity matrix and the potential reads $V = \frac{1}{2}\kappa^{-2}m^2\phi^2$. The mass parameter m is given in units of the reduced Planck mass κ^{-1}. The slow-roll equation of motion for the background fields simplifies to

$$\dot{\boldsymbol{\phi}} = \dot{\phi}\,\hat{\boldsymbol{\phi}} + \phi\,\dot{\hat{\boldsymbol{\phi}}} = -\sqrt{\frac{2}{3}}\,\frac{m}{\kappa^2}\,\hat{\boldsymbol{\phi}} \qquad \Rightarrow \qquad \dot{\phi} = -\sqrt{\frac{2}{3}}\,\frac{m}{\kappa^2} \quad \text{and} \quad \dot{\hat{\boldsymbol{\phi}}} = 0. \tag{104}$$

Here we have used the fact that $\hat{\boldsymbol{\phi}}$ and $\dot{\hat{\boldsymbol{\phi}}}$ are perpendicular, as can be seen by differentiating the relation $\hat{\boldsymbol{\phi}}^T\hat{\boldsymbol{\phi}} = 1$. This means that the direction of $\boldsymbol{\phi}$ is fixed in time; only its magnitude changes. The scalar equation can of course be solved easily, and we obtain

$$\boldsymbol{\phi}(t) = \left(1 - \frac{t}{t_\infty}\right)\boldsymbol{\phi}_0, \qquad \text{with} \qquad t_\infty = \sqrt{\frac{3}{2}}\frac{\kappa^2\phi_0}{m}, \tag{105}$$

where we used the initial condition $\boldsymbol{\phi}(0) = \boldsymbol{\phi}_0$. Here t_∞ is the time when $\boldsymbol{\phi} = 0$ if slow roll would be valid until the end of inflation.

Using this solution, we calculate the Hubble parameter H and the number of e-folds N:

$$H(t) = \frac{m}{\sqrt{6}}\,\phi = \frac{m\phi_0}{\sqrt{6}}\left(1 - \frac{t}{t_\infty}\right), \qquad N(t) = \int_0^t H\mathrm{d}t' = N_\infty - \frac{1}{4}\kappa^2\phi_0^2\left(1 - \frac{t}{t_\infty}\right)^2, \tag{106}$$

where $N_\infty = \frac{1}{4}\kappa^2\phi_0^2$. Next we calculate the slow-roll functions. Since $\boldsymbol{\phi}(t)$ is linear in time, $\tilde{\eta}^{\|}$ and $\tilde{\eta}^{\perp}$ are zero to this order in slow roll. This implies that to this order the decoupling of the Newtonian potential from the scalar field perturbations is exact, see (61). For $\tilde{\epsilon}$ we find

$$\tilde{\epsilon} = (H^{-1})^{\cdot} = \frac{2}{\kappa^2\phi_0^2}\left(1 - \frac{t}{t_\infty}\right)^{-2}. \tag{107}$$

Clearly, $\tilde{\epsilon}$ becomes infinite when $t \to t_\infty$, which is in contradiction with the bound $\tilde{\epsilon} < 3$ derived in subsection 3.3. But of course slow roll has certainly stopped when $t \geq t_1 \equiv t_\infty - \sqrt{3}\,\kappa/m$, because then $\tilde{\epsilon} \geq 1$, so that results obtained from equations valid only within slow roll cannot be trusted. Notice that we do not have a slow-roll period at all if $\phi_0 \leq \sqrt{2}/\kappa$, since then $t_1 \leq 0$, so that $\tilde{\epsilon}$ is never smaller than 1. We regard the quantity N_∞ as the number of e-folds at the end of inflation. This might seem questionable, as t_1 is a better candidate for the end time of (slow-roll) inflation than t_∞. However, the difference in the number of e-folds using t_∞ and t_1 is small: $N_\infty - N(t_1) = \frac{1}{2}$, so that we can safely use N_∞ as a good approximation for the number of e-folds at the end of inflation.

We finish by calculating the model dependent factor $|D_{\mathbf{k}}|^2$, which we need to compute the correlation function of Φ (see section 4). To this end we must determine the time t_k when a mode function $u_{\mathbf{k}}$ changes its behaviour (see section 3.4). We are especially interested in those scales that are observable in the CMBR. As mentioned in the introducion of this paper, the corresponding mode functions change behaviour in a small interval about 60 e-folds before the end of inflation. Hence we consider $N_k \approx 60$ to be a fixed quantity, and determine t_k by means of the definition $N_k = N_\infty - N(t_k)$. We find

$$\frac{t_k}{t_\infty} = 1 - \sqrt{\frac{N_k}{N_\infty}} \qquad \Rightarrow \qquad \tilde{\epsilon}_k = \frac{1}{2N_k}, \tag{108}$$

so that slow roll is still a good approximation at time t_k. To leading order in slow roll we obtain the following expression for $|D_{\mathbf{k}}|^2$:

$$|D_{\mathbf{k}}|^2 = \frac{\kappa^2}{2}\frac{H_k^2}{\tilde{\epsilon}_k} = \frac{2}{3}\, m^2\, N_k^2. \tag{109}$$

5.1.2 Scalar fields with a quadratic potential on a flat manifold

Now we consider a more general symmetric mass matrix $\kappa^{-1}\mathbf{m}$ in the potential. It does not necessarily have to be diagonalized, but because it is symmetric we can always bring it in diagonal form. As a further assumption we take all eigenvalues to be positive, otherwise the potential would not be bounded from below. The potential is denoted by V_2 and given by

$$V_2 = \frac{1}{2}\kappa^{-2}\, \boldsymbol{\phi}^T \mathbf{m}^2 \boldsymbol{\phi}, \tag{110}$$

so that the slow-roll equation of motion (103) reduces to

$$\dot{\boldsymbol{\phi}} = -\sqrt{\frac{2}{3}}\,\frac{1}{\kappa^2}\,\frac{1}{\sqrt{\boldsymbol{\phi}^T \mathbf{m}^2 \boldsymbol{\phi}}}\,\mathbf{m}^2 \boldsymbol{\phi}. \tag{111}$$

The solution of this vector equation can be written in terms of one dimensionless scalar function $\psi(t)$ as

$$\boldsymbol{\phi}(t) = e^{-\frac{1}{2}\mathbf{m}^2 \psi(t)}\boldsymbol{\phi}_0. \tag{112}$$

Here $\boldsymbol{\phi}_0 = \boldsymbol{\phi}(0)$ is the initial starting point of the field $\boldsymbol{\phi}$, which implies that $\psi(0) = 0$. In other words, we have determined the trajectory that the field $\boldsymbol{\phi}$ follows through field space starting from point $\boldsymbol{\phi}_0$.

An important role in our further analyses is played by the functions F_n, defined by

$$F_n = \frac{\boldsymbol{\phi}_0^T \mathbf{m}^{2n} e^{-\mathbf{m}^2 \psi} \boldsymbol{\phi}_0}{\phi_0^2}, \tag{113}$$

with ϕ_0 the length of $\boldsymbol{\phi}_0$: $\phi_0^2 = \boldsymbol{\phi}_0^T \boldsymbol{\phi}_0$. The functions $F_n(\psi)$ are positive and monotonously decreasing for all ψ, tending to zero in the limit $\psi \to \infty$, because we have assumed that all mass eigenvalues are positive. Using these definitions we see that the function $\psi(t)$ is determined by the following equations:

$$\dot{\psi} = \sqrt{\frac{2}{3}} \frac{2}{\kappa^2 \phi_0} \frac{1}{\sqrt{F_1(\psi)}}, \qquad\qquad \psi(0) = 0. \tag{114}$$

Notice that ψ is always non-negative. The functions F_n do not depend on the length of $\boldsymbol{\phi}_0$, only on its direction, as can be seen from the definition (113). This implies that the only dependence on the length ϕ_0 in (114) is in the factor $2/(\kappa^2 \phi_0)$, so that it can be absorbed by a redefinition of the time variable only.

Next we discuss some additional properties of the functions F_n. The definition of F_n can also be written as

$$F_n = \frac{\boldsymbol{\phi}_0^T e^{-\frac{1}{2}\mathbf{m}^2 \psi} \mathbf{m}^{n-p} \mathbf{m}^{n+p} e^{-\frac{1}{2}\mathbf{m}^2 \psi} \boldsymbol{\phi}_0}{\phi_0^2}, \tag{115}$$

for any integer $-n \leq p \leq n$. Using the Green-Schwarz inequality $(\mathbf{A}^T \mathbf{B})^2 \leq (\mathbf{A}^T \mathbf{A})(\mathbf{B}^T \mathbf{B})$ for arbitrary vectors $\mathbf{A}$ and $\mathbf{B}$, we obtain

$${F_n}^2 \leq F_{n-p}\, F_{n+p}. \tag{116}$$

From the definition of the F_n we also see that

$$\frac{\mathrm{d}}{\mathrm{d}\psi} F_n(\psi) = -F_{n+1}(\psi). \tag{117}$$

We can express many important quantities in the functions F_n. The Friedmann equation (103) for H simplifies to

$$H = \frac{\phi_0}{\sqrt{6}} \sqrt{F_1(\psi)}. \tag{118}$$

Formally integrating equation (114) for ψ, we find how long it takes to go from $\psi = 0$ to ψ in slow roll:

$$t(\psi) = \sqrt{\frac{3}{8}} \kappa^2 \phi_0 \int_0^{\psi} \mathrm{d}\psi' \sqrt{F_1(\psi')}. \tag{119}$$

The number of e-folds $N = \int H \mathrm{d}t$ can be interpreted as a function of ψ given by

$$N(\psi) = \frac{\kappa^2 \phi_0^2}{4} \int_0^{\psi} \mathrm{d}\psi' F_1(\psi') = N_\infty \left(1 - F_0(\psi)\right), \tag{120}$$

where we combined (118) and (119), and used (117) to perform the integration. The number of e-folds in the limit $\psi \to \infty$ is given by $N_\infty = \frac{1}{4}\kappa^2 \phi_0^2$, which can be interpreted as an

upper limit for the total number of e-folds during slow-roll inflation. Finally, the slow-roll functions (55) can now be written as

$$\tilde{\epsilon} = \frac{2}{\kappa^2\phi_0^2}\frac{F_2}{F_1^2}, \qquad \tilde{\eta}^{\|} = -\frac{2}{\kappa^2\phi_0^2}\frac{F_3F_1 - F_2^2}{F_1^2F_2}, \qquad \tilde{\eta}^{\perp} = \frac{2}{\kappa^2\phi_0^2}\frac{\sqrt{F_4F_2 - F_3^2}}{F_1F_2}. \tag{121}$$

Using the Green-Schwarz inequality (116) we see that $\tilde{\eta}^{\|}$ is always negative, while $\tilde{\eta}^{\perp}$ is real, as it should be. Observe that if $\mathbf{m}$ is proportional to the identity, the inequality is saturated and $\tilde{\eta}^{\|}$ and $\tilde{\eta}^{\perp}$ are zero. This is in agreement with the results of the previous subsection. Since the functions $F_n(\psi)$ are independent of ϕ_0, this dependence enters only in the prefactors of the expressions for $t(\psi)$, $N(\psi)$ and the slow-roll functions.

Before going on to discuss estimates for the functions F_n, we need to introduce some additional notation. We define a semi-positive definite matrix norm:

$$||\mathbf{A}||^2 = \frac{|\mathbf{A}\boldsymbol{\phi}_0|^2}{\phi_0^2}, \tag{122}$$

for any arbitrary $N \times N$-matrix $\mathbf{A}$. The reason that $||\cdot||$ does not define a regular norm is that $||\mathbf{A}||^2 = 0$ does not imply that $\mathbf{A} = 0$; we can only infer that $\mathbf{A}\boldsymbol{\phi}_0 = 0$. Indeed, if $\mathbf{A}$ has determinant zero and $\boldsymbol{\phi}_0$ is one of $\mathbf{A}$'s zero modes, $\mathbf{A}\boldsymbol{\phi}_0 = 0$ is satisfied without $\mathbf{A}$ being the zero matrix. With this norm the definition of $F_n(\psi)$ can also be written as

$$F_n(\psi) = ||\mathbf{m}^n e^{-\frac{1}{2}\mathbf{m}^2\psi}||^2. \tag{123}$$

We order the eigenvalues of $\mathbf{m}^2$ from smallest to largest, $m_1^2 < m_2^2 < \ldots < m_\ell^2$. Here we look only at distinct eigenvalues, so that ℓ is smaller than N if there are degenerate eigenvalues. The projection operator $\mathbf{E}_n$ projects on the eigenspace with eigenvalue m_n^2. These operators are mutually orthogonal and sum to the identity: $\sum \mathbf{E}_n = \mathbb{1}$. The norm of these projection operators satisfies $||\mathbf{E}_n|| \leq 1$.

Above we have been able to write all kinds of important quantities for the slow-roll period in terms of the functions $F_n(\psi)$. But these functions are rather complicated as they depend both on an (exponentiated) mass matrix $\mathbf{m}^2$ and on the direction of the initial vector $\boldsymbol{\phi}_0$. Since in the CMBR we cannot see further back than about the last 60 e-folds of inflation, while the total amount of inflation is generally much larger, we are often only interested in the asymptotic behaviour of the functions $F_n(\psi)$ for large ψ. Below we will show that the asymptotic behaviour is indeed a good approximation for the time interval during inflation that is (indirectly) observable through the CMBR, but first we concentrate on the asymptotic expressions themselves.

As can be seen from the definition of F_n in (113), in the limit $\psi \to \infty$ the smallest mass eigenvalue will start to dominate. We denote the smallest eigenvalue by μ, $\mu \equiv m_1$, while the ratio of the next-to-smallest and smallest masses squared is called ρ: $\rho \equiv m_2^2/m_1^2 > 1$. Furthermore, the operator $\mathbf{E} \equiv \mathbf{E}_1$ projects on the eigenspace of the smallest eigenvalue,[7]

[7]Here we assumed that $||\mathbf{E}_1|| \neq 0$. If $||\mathbf{E}_1||$ is zero, it means that $\boldsymbol{\phi}_0$ has no component in the subspace corresponding with this eigenvalue. As can be seen from the differential equation for this quadratic case (111), this means that $\boldsymbol{\phi}$ will never obtain a component in the directions corresponding to this subspace. Hence we remove these directions from the problem, consider $\boldsymbol{\phi}$ to be a vector of appropriate (lower) dimension, and take m_1 to be the smallest remaining eigenvalue, etc.

and we define $\chi \equiv ||\mathbf{E}_2||^2/||\mathbf{E}_1||^2$. Using these definitions, we find the following asymptotic behaviour for the functions $F_n(\psi)$ in the limit $\psi \to \infty$:

$$F_n \to ||\mathbf{E}||^2 \mu^{2n} e^{-\mu^2\psi} \left(1 + \chi\rho^n e^{-(\rho-1)\mu^2\psi}\right) \to ||\mathbf{E}||^2 \mu^{2n} e^{-\mu^2\psi}, \tag{124}$$

where the first limit contains both leading and next-to-leading order terms, while the second contains only the leading order term. Both these asymptotic expressions for F_n are needed to obtain the non-vanishing leading order behaviour of ratios and differences of ratios of the functions F_n:

$$\frac{F_p}{F_q} \to \mu^{2(p-q)}, \qquad \frac{F_{n+1}}{F_n} - \frac{F_n}{F_{n-1}} \to \mu^2 \rho^{n-1}(\rho-1)^2 \chi e^{-(\rho-1)\mu^2\psi}. \tag{125}$$

Using these expressions we find the asymptotic behaviour for the Hubble parameter (118) and the number of e-folds (120),

$$H(\psi) \to \frac{\phi_0\,\mu||\mathbf{E}||}{\sqrt{6}}\, e^{-\frac{1}{2}\mu^2\psi}, \qquad N(\psi) \to N_\infty \left(1 - ||\mathbf{E}||^2 e^{-\mu^2\psi}\right). \tag{126}$$

The asymptotic behaviour of the slow-roll functions (121) is given by

$$\begin{aligned} \tilde{\epsilon} &\to \frac{2}{\kappa^2\phi_0^2}\frac{1}{||\mathbf{E}||^2}\, e^{\mu^2\psi}, & \tilde{\eta}^{\|} &\to -\frac{2}{\kappa^2\phi_0^2}\frac{\chi}{||\mathbf{E}||^2}\,\rho(\rho-1)^2 e^{-(\rho-2)\mu^2\psi}, \\ \tilde{\eta}^{\perp} &\to \frac{2}{\kappa^2\phi_0^2}\frac{\sqrt{\chi}}{||\mathbf{E}||^2}\,\rho(\rho-1)e^{-\frac{1}{2}(\rho-3)\mu^2\psi}. \end{aligned} \tag{127}$$

Notice that $\tilde{\eta}^{\|}$ goes to zero for $\rho > 2$, while for $\rho < 2$ it diverges. The same holds true for $\tilde{\eta}^{\perp}$, but there the critical value is $\rho = 3$. Since $\rho > 1$ by definition, the slow-roll function $\tilde{\epsilon}$ always grows faster than $\tilde{\eta}^{\|}$ and $\tilde{\eta}^{\perp}$ in the limit $\psi \to \infty$. As we discussed in section 3.3, however, it is really the combinations $\sqrt{\tilde{\epsilon}}\,\tilde{\eta}^{\|}$ and $\sqrt{\tilde{\epsilon}}\,\tilde{\eta}^{\perp}$ that determine whether slow roll is valid or not. Their critial values are $\rho = 5/2$ and $\rho = 4$, respectively.

We finish this subsection by calculating the model dependent factor $|D_{\mathbf{k}}|^2$. Following the same steps as in the previous subsection we find that $\psi_k = \psi(t_k)$ and $\tilde{\epsilon}_k = \tilde{\epsilon}(t_k)$ are given by

$$e^{-\mu^2\psi_k} = \frac{1}{||\mathbf{E}||^2}\frac{N_k}{N_\infty}, \qquad \tilde{\epsilon}_k = \frac{1}{2N_k}. \tag{128}$$

So as long as $N_\infty \gg N_k$ our assumption of using the asymptotic behaviour for $\psi \to \infty$ at time t_k is very good. Moreover, since in this limit $\tilde{\epsilon}$ is the largest of the three slow-roll functions, we see that the slow-roll approximation is also valid. Hence we can use the leading order slow-roll estimate of (74) for $|D_{\mathbf{k}}|^2$:

$$|D_{\mathbf{k}}|^2 = \frac{\kappa^2}{2}\frac{H_k^2}{\tilde{\epsilon}_k} = \frac{2}{3}\,\mu^2\,N_k^2, \tag{129}$$

where, apart from the previous two expressions, we also used (126) for H. This result agrees with (109) for identical masses.

5.2 Slow roll on a curved manifold

Now we turn to the slow-roll behaviour of scalar fields that parameterize a curved manifold that is isotropic around a point. We start with setting up the general framework, which we clarify by examples and expand upon in the special cases discussed in the next subsections. Consider an N-dimensional manifold with coordinates $\boldsymbol{\phi}$ and metric $\mathbf{G}(\boldsymbol{\phi})$ given by

$$\mathbf{G}(\boldsymbol{\phi}) = g(\boldsymbol{\phi}) \left(\mathbb{1}_N + \frac{\lambda(\boldsymbol{\phi})}{1-\lambda(\boldsymbol{\phi})} \mathbf{Q} \right), \tag{130}$$

with $g(\boldsymbol{\phi}) \neq 0$ and $\lambda(\boldsymbol{\phi}) \neq 1$. The matrix $\mathbf{Q}$ is the projection operator defined by

$$\mathbf{Q} \equiv \mathbf{e}_0 \mathbf{e}_0^T, \qquad \text{with} \qquad \mathbf{e}_0 \equiv \frac{\boldsymbol{\phi}}{\phi}. \tag{131}$$

Here $\phi = \sqrt{\boldsymbol{\phi}^T \boldsymbol{\phi}}$ represents the coordinate length of the vector $\boldsymbol{\phi}$, which should not be confused with $|\boldsymbol{\phi}| = \sqrt{\boldsymbol{\phi}^T \mathbf{G} \boldsymbol{\phi}}$. By taking this form for the metric we have of course restricted ourselves to manifolds that are isotropic around a point, but it covers some general, interesting cases, e.g. the sphere with either embedding (see section 5.2.2) or stereographical coordinates. Also, the equations of motion in a central potential obtained with this metric are identical to those obtained in the case of a more general metric, as we will explain in section 5.2.1. The inverse of this metric and the determinant are given by

$$\mathbf{G}^{-1} = \frac{1}{g} \left(\mathbb{1}_N - \lambda \mathbf{Q} \right), \qquad\qquad \det \mathbf{G} = \frac{g^N}{1-\lambda}. \tag{132}$$

For the determinant we used the relation $\ln \det \mathbf{G} = \operatorname{tr} \ln \mathbf{G}$ and the fact that $\operatorname{tr} \mathbf{Q} = 1$.

Inserting our special choice for the metric into the (pure) slow-roll equation of motion for $\boldsymbol{\phi}$ gives:

$$\dot{\boldsymbol{\phi}} = -\frac{2}{\sqrt{3}\,\kappa} \mathbf{G}^{-1} \boldsymbol{\partial}^T \sqrt{V(\boldsymbol{\phi})} = -\frac{2}{\sqrt{3}} \frac{1}{\kappa g} \left(\boldsymbol{\partial}^T \sqrt{V} - \frac{\lambda}{\phi^2} (\boldsymbol{\partial} \sqrt{V}\, \boldsymbol{\phi}) \boldsymbol{\phi} \right). \tag{133}$$

Notice that $\boldsymbol{\partial}\sqrt{V}\, \boldsymbol{\phi} = \phi^a\, \partial_a \sqrt{V}$ is a scalar. In general this vector equation can be hard to solve, but in practice we often have some information from the corresponding flat case that we can use. In particular, we can often determine the trajectories that the scalar fields follow through the flat field space. On the other hand, it is much harder to calculate exactly how the scalar fields move along these trajectories as a function of time, but this still means that we have reduced the system of N differential equations for $\boldsymbol{\phi}$ to a single one that gives the velocity along the trajectories. In other words, the trajectories of the slow-roll equation of motion for the flat case can be written as

$$\boldsymbol{\phi}_{\text{flat}}(t) = \mathbf{T}(\psi(t), \boldsymbol{\phi}_0), \qquad \text{with} \qquad \mathbf{T}(0, \boldsymbol{\phi}_0) = \boldsymbol{\phi}_0 \qquad \text{and} \qquad \psi(t_0) = 0 \tag{134}$$

(with $\mathbf{T}$ a known function), where the function $\psi(t)$ has to satisfy the differential equation

$$\dot{\psi} = -\frac{2}{\sqrt{3}\,\kappa} \frac{\boldsymbol{\partial}\sqrt{V}\, \mathbf{T}_{,\psi}}{\mathbf{T}_{,\psi}^T \mathbf{T}_{,\psi}} \qquad\qquad \text{(flat case).} \tag{135}$$

An example of this was given in (112) and (114) for the case of a quadratic potential.

This flat solution can be generalized to curved manifolds with a metric of the form introduced above by defining

$$\boldsymbol{\phi}_{\text{curved}}(t) = s(\psi(t))\,\mathbf{T}(\psi(t), \boldsymbol{\phi}_0). \tag{136}$$

Here $\mathbf{T}$ is the same function as above, while the differential equation (135) for ψ is slightly modified to

$$\dot{\psi} = -\frac{2}{\sqrt{3}\kappa}\frac{1}{gs}\frac{\boldsymbol{\partial}\sqrt{V}\,\mathbf{T}_{,\psi}}{\mathbf{T}^T_{,\psi}\mathbf{T}_{,\psi}} \qquad \text{(curved case)}. \tag{137}$$

By inserting our ansatz for the solution into the equation of motion we find that the factor $s(\psi)$ has to satisfy

$$-\frac{s_{,\psi}}{s} = \lambda\,\frac{\mathbf{T}^T_{,\psi}\mathbf{T}_{,\psi}}{\mathbf{T}^T\mathbf{T}}\,\frac{\boldsymbol{\partial}\sqrt{V}\,\mathbf{T}}{\boldsymbol{\partial}\sqrt{V}\,\mathbf{T}_{,\psi}} \qquad \text{and} \qquad s(0) = 1. \tag{138}$$

We give examples of this general method in the following subsections.

Next we discuss the definition and evaluation of the slow-roll functions $\tilde{\epsilon}$, $\tilde{\eta}^{\|}$ and $\tilde{\eta}^{\perp}$. To this end we define functions $C_n(V)$ as follows:

$$C_1(V) = \kappa^2\,\frac{2V}{\phi_0^2}, \qquad C_n(V) = \kappa^{2n}\,\frac{\boldsymbol{\nabla}V(\mathbf{G}^{-1}\boldsymbol{\nabla}^T\boldsymbol{\nabla}V)^{n-2}\mathbf{G}^{-1}\boldsymbol{\nabla}^T V}{\phi_0^2}, \quad n \geq 2. \tag{139}$$

The functions $C_n(V)$ are more than simply the curved generalization of the functions F_n: the C_n are defined for an arbitrary potential, while in the definition of the F_n we have assumed a quadratic potential and made use of the fact that we can determine the trajectories of the fields in that case. Using the Green-Schwarz inequality we can derive the following inequalities for positive integers n, p with $0 < p < n$:

$$C_n^2 \leq C_{2p}C_{2(n-p)}, \tag{140}$$

which follows by writing $\phi_0^2 C_n = \kappa^{2n}\boldsymbol{\nabla}V(\mathbf{G}^{-1}\boldsymbol{\nabla}^T\boldsymbol{\nabla}V)^{p-1}\mathbf{G}^{-1}(\boldsymbol{\nabla}^T\boldsymbol{\nabla}V\mathbf{G}^{-1})^{n-p-1}\boldsymbol{\nabla}^T V$. The slow-roll functions (55) can be written as

$$\tilde{\epsilon} = \frac{2}{\kappa^2\phi_0^2}\frac{C_2}{C_1^2}, \qquad \tilde{\eta}^{\|} = -\frac{2}{\kappa^2\phi_0^2}\frac{C_3C_1 - C_2^2}{C_1^2 C_2}, \qquad (\tilde{\eta}^{\perp})^2 = \left(\frac{2}{\kappa^2\phi_0^2}\right)^2\frac{C_2C_4 - C_3^2}{C_1^2 C_2^2}, \tag{141}$$

which are the same expressions as in the case of a quadratic potential on a flat manifold (121), but with the F_n replaced by $C_n(V)$. The only inequality for the functions C_n that is directly applicable is for $n = 3, p = 1$: $C_3^2 \leq C_2C_4$, which implies that the square $(\tilde{\eta}^{\perp})^2$ is positive, as it should be.

On a curved manifold the second order covariant derivative $\boldsymbol{\nabla}^T\boldsymbol{\nabla}V$ contains connection terms. We now compute what these terms are in the case of our special metric. It is convenient to work out the special combination

$$\mathbf{v} = (\boldsymbol{\nabla}^T\boldsymbol{\nabla}V)\mathbf{G}^{-1}\boldsymbol{\nabla}^T V, \tag{142}$$

since that is how the connection enters into the expressions of the slow-roll functions, as can be seen by writing

$$C_3 = \kappa^6\,\frac{\boldsymbol{\nabla}V\mathbf{G}^{-1}\mathbf{v}}{\phi_0^2}, \qquad C_4 = \kappa^8\,\frac{\mathbf{v}^T\mathbf{G}^{-1}\mathbf{v}}{\phi_0^2}. \tag{143}$$

This vector $\mathbf{v}$ can be split into two vectors $\mathbf{v} = \mathbf{v}_F + \mathbf{v}_G$, given by

$$\mathbf{v}_F = (\boldsymbol{\partial}^T\boldsymbol{\partial}V)\mathbf{G}^{-1}\boldsymbol{\partial}^T V, \qquad (\mathbf{v}_G)_a = -\Gamma^c{}_{ab}\partial_c V(\mathbf{G}^{-1}\boldsymbol{\partial}^T V)^b = \frac{1}{2}(\mathbf{G}^{-1})^{bc}{}_{,a}\partial_b V\partial_c V. \quad (144)$$

Here we have used that $-\Gamma^c{}_{ab}w_c(\mathbf{G}^{-1}\mathbf{w})^b = -\Gamma_{c\,ab}(\mathbf{G}^{-1}\mathbf{w})^b(\mathbf{G}^{-1}\mathbf{w})^c = \frac{1}{2}(\mathbf{G}^{-1})^{bc}{}_{,a}w_b w_c$ for any vector $\mathbf{w}$. Next we compute the derivative of the inverse metric. Using the definitions of the inverse metric $\mathbf{G}^{-1}$ (132) and the projection operator $\mathbf{Q}$ (131), we obtain

$$\frac{1}{2}(\mathbf{G}^{-1})^{bc}{}_{,a} = -\frac{g_{,x}}{g}\frac{\phi_a}{R^2}(\mathbf{G}^{-1})^{bc} - \frac{\lambda_{,x}}{g}\frac{\phi_a}{R^2}\mathbf{Q}^{bc} - \frac{\lambda}{2g}\frac{1}{\phi^2}\left(\delta^b_a\phi^c + \delta^c_a\phi^b - 2\frac{\phi^b\phi^c}{\phi^2}\phi_a\right). \quad (145)$$

Here we have assumed that the metric functions g and λ only depend implicitly on $\boldsymbol{\phi}$ via the quantity $x \equiv \phi^2/R^2$, with R the characteristic radius of curvature of the manifold. This leads to the final result for $\mathbf{v}_G$:

$$\mathbf{v}_G = -\frac{1}{gR^2}\left[g_{,x}\,\boldsymbol{\partial}V\,\mathbf{G}^{-1}\boldsymbol{\partial}^T V + \frac{\lambda}{\phi^2}\Big(\frac{\lambda_{,x}}{\lambda} - \frac{R^2}{\phi^2}\Big)(\boldsymbol{\partial}V\,\boldsymbol{\phi})^2\right]\boldsymbol{\phi} - \frac{\lambda}{g}(\boldsymbol{\partial}V\boldsymbol{\phi})\frac{\boldsymbol{\partial}^T V}{\phi^2}. \quad (146)$$

In the next subsections we work out these expressions for the slow-roll functions in the cases of some special potentials.

5.2.1 Scalar fields on a curved manifold with a central potential

We consider the case that the potential $V_c(\boldsymbol{\phi})$ is a central potential around the origin: it is a function of the coordinate length ϕ only. The first and second order gradients $\boldsymbol{\partial}^T V_c$ and $\boldsymbol{\partial}^T\boldsymbol{\partial}V_c$ are then given by

$$\boldsymbol{\partial}^T V_c = V_{c,\phi}\mathbf{e}_0 \qquad \text{and} \qquad \boldsymbol{\partial}^T\boldsymbol{\partial}V_c = V_{c,\phi\phi}\mathbf{Q} + \frac{1}{\phi}V_{c,\phi}(\mathbb{1} - \mathbf{Q}). \quad (147)$$

As in our first example of scalar fields with equal masses on a flat manifold in section 5.1.1, we find that the vector slow-roll equation of motion reduces to a scalar equation:

$$\dot{\boldsymbol{\phi}} = -\frac{1}{\sqrt{3}\kappa}\frac{1-\lambda}{g}\frac{V_{c,\phi}}{\sqrt{V_c}}\,\mathbf{e}_0 \qquad \Rightarrow \qquad \dot{\phi} = -\frac{1}{\sqrt{3}\kappa}\frac{1-\lambda}{g}\frac{V_{c,\phi}}{\sqrt{V_c}}. \quad (148)$$

Notice that if we take a more general metric

$$\tilde{\mathbf{G}}(\boldsymbol{\phi}) = g(\boldsymbol{\phi})\left(\mathbb{1}_N + \sum_{n=0}^{N-1}\frac{\lambda_n(\boldsymbol{\phi})}{1-\lambda_n(\boldsymbol{\phi})}\mathbf{Q}_n\right), \quad (149)$$

with a sum of mutually orthogonal projectors $\mathbf{Q}_n$, instead of just the one $\mathbf{Q} = \mathbf{Q}_0$, no other terms appear. This is because $\boldsymbol{\partial}V_c$ is pointing in the radial direction $\mathbf{e}_0$ and $\mathbf{e}_0$ is an eigenvector of the metric $\tilde{\mathbf{G}}$ as well as of metric $\mathbf{G}$: $\tilde{\mathbf{G}}\mathbf{e}_0 = \mathbf{G}\mathbf{e}_0 = \frac{g}{1-\lambda}\mathbf{e}_0$.

Before going on to discuss the slow-roll functions to leading order in slow roll, we note that we can also say something about the exact slow-roll function $\tilde{\eta}^\perp$. From the exact definition of $\tilde{\boldsymbol{\eta}}$ in (48) and the exact equation of motion (47) we get

$$(\tilde{\eta}^\perp)^2 = \frac{\boldsymbol{\partial}V(\mathbb{1} - \mathbf{e}_1\mathbf{e}_1^T\mathbf{G})\mathbf{G}^{-1}\boldsymbol{\partial}^T V}{H^2|\dot{\boldsymbol{\phi}}|^2} = \frac{1-\lambda}{g}\left(\frac{V_{c,\phi}}{H|\dot{\boldsymbol{\phi}}|}\right)^2\left(1 - \cos^2\alpha\right), \quad (150)$$

where α denotes the angle between the unit vectors $\mathbf{e}_0$ and $\mathbf{e}_1$, or equivalently, between the position vector $\boldsymbol{\phi}$ and the velocity $\dot{\boldsymbol{\phi}}$. This angle α is defined in curved field space as

$$\cos\alpha = \frac{\mathbf{e}_1 \cdot \mathbf{e}_0}{|\mathbf{e}_1||\mathbf{e}_0|} = \sqrt{\frac{g}{1-\lambda}}\, \mathbf{e}_1^T \mathbf{e}_0. \tag{151}$$

Notice that while $|\mathbf{e}_1| = 1$, this is not the case for $|\mathbf{e}_0|$, as $\mathbf{e}_0$ was defined as having coordinate length equal to one. From this formula we infer that if the field velocity $\dot{\boldsymbol{\phi}}$ is pointing in the same direction (up to orientation) as the coordinate vector $\boldsymbol{\phi}$ (i.e. $\alpha = 0, \pi$), the slow-roll function $\tilde{\eta}^{\perp}$ vanishes. Notice that if this happens, it holds for all time, as can be seen from the exact equation of motion.

Next we work out the expressions for the slow-roll functions to leading order in slow roll, given in (141). As discussed in the previous section, it is useful to first calculate the vector $\mathbf{v} = \mathbf{v}_F + \mathbf{v}_G$, defined in (142) ff.. We find in the case of a central potential

$$\mathbf{v} = \frac{1-\lambda}{g} V_{c,\phi} \left(V_{c,\phi\phi} - \frac{g_{,x}}{g}\frac{\phi}{R^2} V_{c,\phi} \right) \mathbf{e}_0 - \frac{\lambda_{,x}}{g}\frac{\phi}{R^2} (V_{c,\phi})^2 \mathbf{e}_0, \tag{152}$$

which leads to

$$\tilde{\epsilon} = \frac{1-\lambda}{2g}\left(\frac{V_{c,\phi}}{\kappa V_c}\right)^2, \qquad \tilde{\eta}^{\|} - \tilde{\epsilon} = -\frac{1-\lambda}{g}\left(\frac{V_{c,\phi\phi}}{\kappa^2 V_c} - \frac{g_{,x}}{g}\frac{\phi}{R^2}\frac{V_{c,\phi}}{\kappa^2 V_c}\right) + \frac{\lambda_{,x}}{g}\frac{\phi}{R^2}\frac{V_{c,\phi}}{\kappa^2 V_c}. \tag{153}$$

Furthermore, $\tilde{\eta}^{\perp} = 0$, as can most easily be understood by realizing that the Green-Schwarz inequality is saturated ($C_3^2 = C_2 C_4$) if $\mathbf{v}$ and $\boldsymbol{\nabla} V$ are (anti-)parallel. This is in agreement with the result (150), since to leading order in slow roll $\cos^2\alpha = 1$.

To illustrate various aspects of the general discussion above, we now turn to an example: a quadratic central potential of scalar fields with identical masses $\kappa^{-1}m$, which are the local embedding coordinates on an N-dimensional sphere with radius R. These coordinates $\boldsymbol{\phi}$ are induced by embedding the sphere in an $N+1$ dimensional Euclidean space, so that by construction $\phi^2 = \boldsymbol{\phi}^T\boldsymbol{\phi} < R^2$. At least two of these coordinate systems are needed to cover the whole sphere. In the slow roll discussion here, we stay within one coordinate system because the quadratic potential is minimal in the origin of this system. The metric, its inverse, and the connection are given by

$$\mathbf{G} = \mathbb{1} + \frac{\phi^2}{R^2 - \phi^2}\,\mathbf{Q}, \qquad \mathbf{G}^{-1} = \mathbb{1} - \frac{\phi^2}{R^2}\,\mathbf{Q}, \qquad \text{and} \qquad \Gamma^a_{\ bc} = \frac{\phi^a}{R^2}\, G_{bc}. \tag{154}$$

Hence $g = 1$ and $\lambda = x = \phi^2/R^2 < 1$ in terms of the general metric (130).

Inserting the relevant quantities into the slow-roll equation of motion (148) we find

$$\dot{\phi} = -\sqrt{\frac{2}{3}}\frac{m}{\kappa^2}\left(1 - \frac{\phi^2}{R^2}\right). \tag{155}$$

Solving this equation, multiplying by a constant unit vector, and applying the initial condition $\boldsymbol{\phi}(0) = \boldsymbol{\phi}_0$, we get the following answer for the solution of the background equation to leading order in slow roll:

$$\boldsymbol{\phi}(t) = R\, \tanh[\gamma(t_\infty - t)]\, \hat{\boldsymbol{\phi}}_0, \tag{156}$$

with $\gamma = \sqrt{\frac{2}{3}}\,\frac{m}{\kappa^2 R}$ and $t_\infty = \frac{1}{2\gamma}\ln\frac{R+\phi_0}{R-\phi_0}$. The slow-roll functions follow immediately from (153):

$$\tilde{\epsilon} = \frac{2}{\kappa^2 R^2}\,\frac{R^2-\phi_0^2}{\phi_0^2}\left(\frac{\sinh(\gamma t_\infty)}{\sinh[\gamma(t_\infty - t)]}\right)^2, \qquad \tilde{\eta}^{\|} = \frac{2}{\kappa^2 R^2}, \qquad \tilde{\eta}^{\perp} = 0. \tag{157}$$

Since R is fixed by the model, the slow roll conditions, $\tilde{\epsilon}$ and $\sqrt{\tilde{\epsilon}}\,\tilde{\eta}^{\|}$ small, are satisfied if ϕ_0 is such that $R^2 - \phi_0^2 \ll R^2$. From the Hubble parameter $H = \kappa\sqrt{V/3} = m\phi/\sqrt{6}$ we derive the expression for the number of e-folds:

$$N(t) = \int_0^t H\mathrm{d}t = N_\infty - \frac{\kappa^2 R^2}{2}\ln\cosh[\gamma(t_\infty - t)], \tag{158}$$

with $N_\infty = -\frac{1}{4}\kappa^2 R^2 \ln(1-\phi_0^2/R^2) \geq 0$, since $0 \leq \phi_0^2 < R^2$. Therefore, the field should start close to the equator of the sphere, $R^2 - \phi_0^2 \ll R^2$, to ensure that sufficient inflation is obtained. This is compatible with the requirement that the relevant slow roll functions ($\tilde{\epsilon}$ and $\sqrt{\tilde{\epsilon}}\,\tilde{\eta}^{\|}$) are small initially. In the limit $R \to \infty$ all results agree with those we found in the flat case in section 5.1.1. Notice that we can also determine the solution for $\boldsymbol{\phi}$ by using our knowledge from the flat case and the method described in (136) ff., but in this particular case that is more complicated.

We finish with the calculation of the expression for the coefficient $|D_{\mathbf{k}}|^2$, following the steps outlined in section 5.1.1. The time t_k and the slow-roll function $\tilde{\epsilon}$ at t_k are given by

$$\cosh[\gamma(t_\infty - t_k)] = \exp\left(\frac{2N_k}{\kappa^2 R^2}\right) \qquad \Rightarrow \qquad \tilde{\epsilon}_k = \frac{2}{\kappa^2 R^2}\,\frac{1}{\exp\left(\frac{4N_k}{\kappa^2 R^2}\right) - 1}. \tag{159}$$

This confirms that the slow-roll approximation is still valid at t_k. For the model dependent factor $|D_{\mathbf{k}}|^2$ we obtain to leading order in slow roll

$$|D_{\mathbf{k}}|^2 = \frac{2}{3}\,m^2\left(\frac{\kappa^2 R^2}{2}\sinh\frac{2N_k}{\kappa^2 R^2}\right)^2. \tag{160}$$

In this calculation we have assumed that $\tilde{\eta}^{\|}$ itself is small, which means that R is larger than the Planck radius κ^{-1}. If this is not the case, more terms in (74) have to be taken into account. In the limit that $R \to \infty$ this result agrees with the flat case (109).

5.2.2 Scalar fields with different masses on a curved manifold

Next we consider the case of the quadratic potential V_2, defined in (110), on a curved manifold with metric (130). The slow-roll equation of motion in this situation reads

$$\dot{\boldsymbol{\phi}} = -\sqrt{\frac{2}{3}}\,\frac{1}{g\kappa^2\sqrt{\boldsymbol{\phi}^T\mathbf{m}^2\boldsymbol{\phi}}}\left(\mathbf{m}^2 - \lambda\frac{\boldsymbol{\phi}^T\mathbf{m}^2\boldsymbol{\phi}}{\phi^2}\mathbb{1}\right)\boldsymbol{\phi}. \tag{161}$$

Since we know the trajectories of $\boldsymbol{\phi}$ in the flat field case (112), we can use the method described in (136) ff. to determine the solution, or at least the trajectories, of $\boldsymbol{\phi}$ in the curved field space. The equation for $s(\psi)$ can be solved analytically for the sphere with embedding coordinates, which was also considered in the previous subsection.

Before turning to this example, we first work out the expressions for the functions $C_1(V_2), \ldots, C_4(V_2)$ in terms of the flat functions F_n defined in (113). These expressions can be used to determine the slow-roll functions (141). Because we found many properties and estimates for the F_n in section 5.1.2, a lot of additional properties are obtained for the functions $C_n(V_2)$ in this way. To find the relation between the $C_n(V_2)$ and the F_n it is convenient to define intermediate functions $\tilde{F}_n$ that incorporate some of the non-flat metric aspects, but not the full covariant derivatives (connection terms). They are defined as follows:

$$\tilde{F}_n = \frac{\boldsymbol{\phi}^T \mathbf{G}(\mathbf{G}^{-1}\mathbf{m}^2)^n \boldsymbol{\phi}}{\phi_0^2}, \qquad n \geq 1. \tag{162}$$

Notice that apart from the metric aspects, the $\tilde{F}_n$ also contain an extra factor of s^2 as compared to the F_n, since $\boldsymbol{\phi} = s(\psi) e^{-\frac{1}{2}\mathbf{m}^2\psi} \boldsymbol{\phi}_0$ according to (136). The functions $\tilde{F}_n$ can be expressed in terms of the F_n; for the first four functions $\tilde{F}_n$ we find by inserting the definition (132) of $\mathbf{G}^{-1}$

$$\tilde{F}_1 = s^2 F_1, \qquad \tilde{F}_2 = \frac{s^2}{g}\left(F_2 - \Lambda F_1\right), \qquad \tilde{F}_3 = \frac{s^2}{g^2}\left(F_3 - 2\Lambda F_2 + \Lambda^2 F_1\right),$$
$$\tilde{F}_4 = \frac{s^2}{g^3}\left(F_4 - \Lambda\left(2F_3 + \frac{F_2^2}{F_1}\right) + 3\Lambda^2 F_2 - \Lambda^3 F_1\right), \qquad \text{with} \qquad \Lambda = \lambda \frac{F_1}{F_0}. \tag{163}$$

Here we used that $\boldsymbol{\phi}^T \mathbf{m}^{2p} \mathbf{Q} \mathbf{m}^{2q} \mathbf{Q} \ldots \mathbf{m}^{2r} \boldsymbol{\phi} = (\frac{F_p}{F_0}\frac{F_q}{F_0} \ldots \frac{F_r}{F_0}) F_0 s^2$. Writing the functions $\tilde{F}_n$ as $\tilde{F}_n = \boldsymbol{\phi}^T(\mathbf{m}^2\mathbf{G}^{-1})^p \mathbf{G}(\mathbf{G}^{-1}\mathbf{m}^2)^{n-p}\boldsymbol{\phi}/\phi_0^2$, we obtain the same Green-Schwarz inequalities as for the C_n:

$$(\tilde{F}_n)^2 \leq \tilde{F}_{2p}\tilde{F}_{2(n-p)} \tag{164}$$

for integer $0 < p < n$. The next (and final) step is to write the functions $C_n(V_2)$ in terms of the $\tilde{F}_n$. It is easy to show that

$$C_1(V_2) = \tilde{F}_1 = s^2 F_1 \qquad \text{and} \qquad C_2(V_2) = \tilde{F}_2 = \frac{s^2}{g}\left(F_2 - \Lambda F_1\right). \tag{165}$$

For the functions C_3 and C_4 we use the vector $\mathbf{v}$ defined in (142), which in the case of a quadratic potential can be written as

$$\mathbf{v} = \frac{1}{\kappa^4}\left(\mathbf{m}^2\mathbf{G}^{-1}\mathbf{m}^2 - \frac{\Lambda}{g}\mathbf{m}^2 - \frac{H}{g}\mathbb{1}\right)\boldsymbol{\phi}, \tag{166}$$

where $H = g_{,x}\frac{\phi_0^2}{R^2}\tilde{F}_2 + \Lambda\tilde{F}_1\left(\frac{\lambda_{,x}}{\lambda}\frac{\phi_0^2}{R^2} - \frac{1}{s^2 F_0}\right)$. By inserting this into the expressions for C_3 and C_4 we obtain

$$C_3(V_2) = \tilde{F}_3 - \frac{\Lambda}{g}\tilde{F}_2 - \frac{1-\lambda}{g}\frac{H}{g}\tilde{F}_1, \tag{167}$$
$$C_4(V_2) = \tilde{F}_4 - 2\frac{\Lambda}{g}\tilde{F}_3 + \left(\frac{\Lambda}{g}\right)^2\tilde{F}_2 - 2\frac{1-\lambda}{g}\frac{H}{g}\left[\tilde{F}_2 - \frac{\Lambda}{g}\tilde{F}_1 - \frac{H}{2g}s^2 F_0\right].$$

Next we discuss the example mentioned above. We consider the case where the manifold on which the fields live is a sphere with radius R. We again use embedding coordinates, i.e. $g = 1$ and $\lambda = x = \phi^2/R^2$. Solving equation (138) for s we find

$$\frac{s_{,\psi}}{s^3} = \frac{\phi_0^2}{2R^2} F_1(\psi) \qquad \Rightarrow \qquad \frac{1}{s^2(\psi)} = 1 - \frac{\phi_0^2}{R^2}\left(1 - F_0(\psi)\right). \tag{168}$$

The trajectories of $\boldsymbol{\phi}$ and the differential equation for ψ (137) are given by

$$\boldsymbol{\phi}(\psi) = s(\psi) e^{-\frac{1}{2}\mathbf{m}^2\psi}\boldsymbol{\phi}_0, \qquad \dot{\psi} = \sqrt{\frac{2}{3}}\frac{2}{\kappa^2\phi_0}\frac{1}{s(\psi)}\frac{1}{\sqrt{F_1(\psi)}}. \tag{169}$$

As in the flat case, ψ is monotonously increasing, starting at zero on $t = 0$, and reaching ∞ when $\boldsymbol{\phi} = 0$. The slow-roll functions are again determined from (141), using the expressions for the C_n derived above. However, since the resulting expressions are quite large, we only give $\tilde{\epsilon}$ here:

$$\tilde{\epsilon} \;=\; \frac{2}{\kappa^2\phi_0^2}\frac{1}{s^2}\frac{F_2}{F_1^2} - \frac{2}{\kappa^2R^2} \;=\; \frac{2}{\kappa^2\phi_0^2}\left(1 - \frac{\phi_0^2}{R^2}\right)\frac{F_2}{F_1^2} + \frac{2}{\kappa^2R^2}\frac{F_0F_2 - F_1^2}{F_1^2}, \tag{170}$$

which, using the Green-Schwarz inequality (116), is seen to be larger than or equal to zero, as it should. For the number of e-folds we find

$$N = \frac{\kappa^2\phi_0^2}{4}\int_0^\psi s^2 F_1 \mathrm{d}\psi = N_\infty - \frac{1}{4}\kappa^2R^2\ln\left(1 + \frac{\phi_0^2}{R^2 - \phi_0^2}F_0(\psi)\right), \tag{171}$$

where N_∞ is the same as in the previous subsection: $N_\infty = \frac{1}{4}\kappa^2R^2\ln(R^2/(R^2 - \phi_0^2))$.

As mentioned above the functions F_n satisfy the estimates (124). This means that we can give the following estimates for H, $\tilde{\epsilon}$ and N in the limit $\psi \to \infty$, using that $\phi_0 < R$:

$$\begin{aligned} &H \to \frac{\phi_0\mu||\mathbf{E}||}{\sqrt{6}}\left[1 - \frac{\phi_0^2}{R^2}\left(1 - ||\mathbf{E}||^2 e^{-\mu^2\psi}\right)\right]^{-\frac{1}{2}} e^{-\frac{1}{2}\mu^2\psi}, \qquad \tilde{\epsilon} \to \frac{2}{\kappa^2\phi_0^2}\frac{1}{||\mathbf{E}||^2}\frac{R^2 - \phi_0^2}{R^2}e^{\mu^2\psi}, \\ &N \to N_\infty - \frac{1}{4}\kappa^2R^2\ln\left(1 + \frac{\phi_0^2}{R^2 - \phi_0^2}||\mathbf{E}||^2 e^{-\mu^2\psi}\right). \end{aligned} \tag{172}$$

Following the same steps as in the previous subsection to determine $|D_\mathbf{k}|^2$, we find for the critical value ψ_k

$$e^{-\mu^2\psi_k} = \frac{1}{||\mathbf{E}||^2}\frac{R^2 - \phi_0^2}{\phi_0^2}\left(\exp\left(\frac{4N_k}{\kappa^2R^2}\right) - 1\right) = \frac{1}{||\mathbf{E}||^2}\frac{\exp\left(\frac{4N_k}{\kappa^2R^2}\right) - 1}{\exp\left(\frac{4N_\infty}{\kappa^2R^2}\right) - 1}. \tag{173}$$

Hence, as in the flat case, the assumption of looking at the asymptotic behaviour for ψ at time t_k is a good approximation if $N_\infty \gg N_k$. Finally, the expression for $|D_\mathbf{k}|^2$ in this limit is

$$|D_\mathbf{k}|^2 = \frac{2}{3}\mu^2\left(\frac{\kappa^2R^2}{2}\sinh\frac{2N_k}{\kappa^2R^2}\right)^2, \tag{174}$$

in agreement with (160) for identical masses.

6 Conclusions

In this paper we have analyzed scalar gravitational perturbations on a Robertson-Walker background in the presence of multiple scalar fields that take values on a (geometrically non-trivial) field manifold during slow-roll inflation.

We have modified the definitions of the well-known slow-roll parameters to define slow-roll functions in terms of the Hubble parameter, background field velocity and their derivatives in the case of multiple scalar field inflation. This means that the slow-roll functions have become vectors, except for $\tilde{\epsilon}$ which is a derivative of the Hubble parameter. Like other relevant vectors they are split into a component parallel to the scalar field velocity, and components perpendicular to this velocity vector. To define the slow-roll functions we do not need to make the assumption that slow roll is valid, but if it is valid one can expand in these functions, giving the relative importance of terms in various equations. For example, if $\tilde{\epsilon}$, $\sqrt{\tilde{\epsilon}}\,\tilde{\eta}^{\perp}$ and $\sqrt{\tilde{\epsilon}}\,\tilde{\eta}^{\|}$ are small, the background equation of motion for the scalar fields can be approximated by the (pure) slow-roll equation.

We set up the combined system of gravitational and matter perturbations in a way analogous to Mukhanov et al. [19], but including multiple scalar fields and effects of a non-trivial field geometry. The component of the scalar field perturbations parallel to the background field velocity can be eliminated. The remaining perpendicular components of the field perturbations and the gravitational potential are described by coupled differential equations (61). However, the gravitational potential decouples from the perpendicular field perturbations if effects of the order of $\sqrt{\tilde{\epsilon}}\,\tilde{\eta}^{\perp}$ can be neglected; to the same order the background equations reduce to the slow-roll equations.

Since to first order in slow roll the equation for the gravitational potential is equal to the single field case, it has the same expressions for the solution as [19] in the small and long wavelength limits. Using the slow-roll functions the corrections due to the transition region between these two limits can be estimated. It follows that simply joining the two solutions together yields a good approximation during the complete inflationary period, in the sense that corrections are of higher order in slow roll. The transition between the two solutions happens when $k^2 = |\theta''/\theta| \approx \mathcal{H}^2(2\tilde{\epsilon} + \tilde{\eta}^{\|})$. Even though this is unequal to the time of horizon crossing $k^2 = \mathcal{H}^2$ of a scale k, the corrections to the gravitational potential if one would use this time instead are suppressed in slow roll.

The quantum two-point correlation function of the gravitational potential is related to the temperature fluctuations that are observed in the CMBR. The only physical degrees of freedom that can be quantized in the system we consider are the scalar field perturbations. (Only the graviton states are physical degrees of freedom on the gravitational side. However, they are decoupled because the Einstein equations have been linearized.) Although the quantization at the beginning of inflation involves the scalar field perturbations, after the Fock space has been constructed the time evolution of the correlator can be calculated as if it is a classical quantity. Choosing the vacuum as the initial state of inflation is a good assumption for the scales that are observable in the CMBR, since their momenta in the initial stages of inflation are much larger than the Planck energy. Taking a thermal state with the Planck temperature at the beginning of inflation as a first attempt at improving on the assumption of a pure vacuum leads to corrections to the vacuum correlator of the order of only 10^{-3} or less. Here we assumed that at least a few e-folds of inflation occured between the Planck time and the moment when the observable scales went through the horizon. The gravitational correlation function contains one directly model dependent factor that can be

expressed in terms of the slow-roll functions.

Finally, we discussed some multiple field examples to illustrate some dynamical aspects of the background and compute this model dependent factor. On flat manifolds with central or quadratic potentials the trajectories of the fields in field space can be found in terms of one function $\psi(t)$, so that only a single differential equation remains. This equation of motion, as well as many other relevant quantities like the slow-roll functions and the Hubble parameter, can be expressed in functions F_n of this ψ. These functions have many useful properties which make it possible to analyze slow-roll phenomena without having to explicitly solve the (complicated) equation for ψ. They satisfy inequalities that follow from the Green-Schwarz inequality. Their asymptotic behaviour for large ψ can be used to analyze various important quantities when observable scales go through the horizon, provided that there have been many e-folds of inflation before that. Using that the derivative of F_n equals $-F_{n+1}$ makes it possible to integrate the Hubble parameter and obtain the number of e-folds as a function of ψ.

The generalizations C_n of the functions F_n to curved field space and to an arbitrary potential lead to the same expressions for the slow-roll functions as in the flat case with a quadratic potential. For such a potential on a manifold isotropic around the origin we expressed the C_n in terms of their flat relatives. If the trajectories of the fields in the flat field space are known for a given potential, the corresponding trajectories on an isotropic manifold are the trajectories in flat space multiplied by a scalar function s of the parameter ψ. The sphere with embedding coordinates is a special example of an isotropic manifold. The background equations can be solved explicitly as a function of time for a quadratic potential with all masses equal. If not all masses are equal, it is still possible to find an integrated expression for the number of e-folds in terms of $F_0(\psi)$. For the flat space examples we considered, the only possibility to obtain a large total number of e-folds is by taking large initial field values. The radius of curvature of a curved manifold is an additional parameter that influences the total number of e-folds: for the examples of the sphere this number becomes large if the initial field values are of the same order as the radius of the sphere.

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A Derivation of the equation of motion for the perpendicular field perturbations

This appendix is devoted to the derivation of the equation of motion for the perpendicular field components $\boldsymbol{\delta}\mathbf{v} = \frac{a}{\kappa^2|\boldsymbol{\phi}'|}\boldsymbol{\delta\phi}^{\perp}$. We start by taking the perpendicular projection of equation (45). Using the antisymmetric properties of the Riemann tensor and the fact that we contract with two identical vectors $\boldsymbol{\phi}'$ we conclude that $\mathbf{P}^{\perp}\mathbf{R}(\boldsymbol{\phi}',\boldsymbol{\phi}')\boldsymbol{\delta\phi} = \mathbf{R}(\boldsymbol{\phi}',\boldsymbol{\phi}')\boldsymbol{\delta\phi}^{\perp}$. The perpendicularly projected equation of motion for $\boldsymbol{\delta\phi}$ can be written as

$$\mathbf{P}^{\perp}\left(\mathcal{D}_\eta^2 + 2\mathcal{H}\mathcal{D}_\eta + a^2\mathbf{M}^2\right)\boldsymbol{\delta\phi}^{\|} + 2a^2\Phi\mathbf{P}^{\perp}\mathbf{G}^{-1}\boldsymbol{\nabla}^T V + \mathbf{P}^{\perp}\left(\mathcal{D}_\eta^2 + 2\mathcal{H}\mathcal{D}_\eta\right)\boldsymbol{\delta\phi}^{\perp} + \Big(a^2(\mathbf{M}^2)^{\perp\perp} - \Delta - \mathbf{R}(\boldsymbol{\phi}',\boldsymbol{\phi}')\Big)\boldsymbol{\delta\phi}^{\perp} = 0. \quad (175)$$

Here we split $\boldsymbol{\delta\phi} = \boldsymbol{\delta\phi}^{\|} + \boldsymbol{\delta\phi}^{\perp}$ into components parallel and perpendicular to $\boldsymbol{\phi}'$. Furthermore, we used that the projection operator commutes with Δ and employed the notation defined in (36). The last term can be trivially rewritten, so we now derive expressions for the first three terms on the left-hand side of (175) in terms of u and $\boldsymbol{\delta}\mathbf{v}$ only.

We consider the first term in (175). This term does not vanish in general, since the derivatives and mass matrix generally change the direction of $\boldsymbol{\delta\phi}^{\|}$. The covariant derivative $\mathcal{D}_\eta^2$ applied to the non-contracted vector $\boldsymbol{\phi}'$ in $\boldsymbol{\delta\phi}^{\|} = \frac{\boldsymbol{\phi}'\cdot\boldsymbol{\delta\phi}}{|\boldsymbol{\phi}'|^2}\boldsymbol{\phi}'$ is canceled by the mass term. This can be seen from the covariantly differentiated background equation of motion (42),

$$\mathcal{D}_\eta^2\boldsymbol{\phi}' + 2\left(\mathcal{H}' - 2\mathcal{H}^2\right)\boldsymbol{\phi}' + a^2\mathbf{M}^2\boldsymbol{\phi}' = 0, \quad (176)$$

where we used $\mathcal{D}_\eta(\mathbf{G}^{-1}\boldsymbol{\nabla}^T V) = \mathbf{M}^2\boldsymbol{\phi}'$, if we apply the perpendicular projection operator: $\mathbf{P}^{\perp}\mathcal{D}_\eta^2\boldsymbol{\phi}' + \mathbf{P}^{\perp}(a^2\mathbf{M}^2\boldsymbol{\phi}') = 0$. The remaining contribution of the first term of (175) can be written as

$$\mathbf{P}^{\perp}\left(\mathcal{D}_\eta^2 + 2\mathcal{H}\mathcal{D}_\eta + a^2\mathbf{M}^2\right)\boldsymbol{\delta\phi}^{\|} = \frac{\kappa^2|\boldsymbol{\phi}'|}{a}\,\frac{2}{\kappa^2}\left[a\,\frac{\boldsymbol{\phi}'\cdot\boldsymbol{\delta\phi}}{|\boldsymbol{\phi}'|^2}\right]'\,\frac{(\mathcal{D}_\eta\boldsymbol{\phi}')^{\perp}}{|\boldsymbol{\phi}'|}. \quad (177)$$

The factor involving $\boldsymbol{\phi}'\cdot\boldsymbol{\delta\phi}$ can be rewritten in terms of u using the (0i)-component of the Einstein equation (41):

$$\frac{2}{\kappa^2}\left[a\,\frac{\boldsymbol{\phi}'\cdot\boldsymbol{\delta\phi}}{|\boldsymbol{\phi}'|^2}\right]' = \frac{2}{\kappa^2}\left[\frac{2\kappa}{|\boldsymbol{\phi}'|}\Big(u' + \frac{|\boldsymbol{\phi}'|'}{|\boldsymbol{\phi}'|}u\Big)\right]' = \frac{4}{\mathcal{H}\sqrt{2\tilde{\epsilon}}}\left[u'' + \left(\frac{|\boldsymbol{\phi}'|''}{|\boldsymbol{\phi}'|} - 2\Big(\frac{|\boldsymbol{\phi}'|'}{|\boldsymbol{\phi}'|}\Big)^2\right)u\right], \quad (178)$$

where we have used that $\kappa|\boldsymbol{\phi}'| = \mathcal{H}\sqrt{2\tilde{\epsilon}}$, which follows from (59) and (53).

For the second term in (175) we use the background field equation (42) and obtain

$$2a^2\Phi\,\mathbf{P}^{\perp}\mathbf{G}^{-1}\boldsymbol{\nabla}^T V = -2\Phi\left(\mathcal{D}_\eta\boldsymbol{\phi}'\right)^{\perp} = \frac{\kappa^2|\boldsymbol{\phi}'|}{a}\,\frac{4}{\mathcal{H}\sqrt{2\tilde{\epsilon}}}(\mathcal{H}' - \mathcal{H}^2)u\,\frac{(\mathcal{D}_\eta\boldsymbol{\phi}')^{\perp}}{|\boldsymbol{\phi}'|}, \quad (179)$$

using another, but equivalent expression for $\kappa|\boldsymbol{\phi}'|$: $\kappa|\boldsymbol{\phi}'| = 2(\mathcal{H}^2 - \mathcal{H}')/(\mathcal{H}\sqrt{2\tilde{\epsilon}})$. Combining these expressions and using the equation of motion for u (58) we find that the first two terms of (175) can be written as

$$\mathbf{P}^{\perp}\left(\ldots\right)\boldsymbol{\delta\phi}^{\|} + 2a^2\Phi\mathbf{P}^{\perp}\mathbf{G}^{-1}\boldsymbol{\nabla}^T V = 4\,\frac{\kappa^2|\boldsymbol{\phi}'|}{a}\left[\frac{\Delta u}{\mathcal{H}\sqrt{2\tilde{\epsilon}}} + \frac{(\mathcal{D}_\eta\boldsymbol{\phi}')^{\perp}}{|\boldsymbol{\phi}'|}\cdot\boldsymbol{\delta}\mathbf{v}\right]\frac{(\mathcal{D}_\eta\boldsymbol{\phi}')^{\perp}}{|\boldsymbol{\phi}'|}. \quad (180)$$

Moving to the third term of (175), we first note that substitution of $\boldsymbol{\delta\phi}^\perp = \frac{\kappa^2|\boldsymbol{\phi}'|}{a}\boldsymbol{\delta}\mathbf{v}$ gives

$$\mathbf{P}^\perp(\mathcal{D}_\eta^2 + 2\mathcal{H}\mathcal{D}_\eta)\boldsymbol{\delta\phi}^\perp = \frac{\kappa^2|\boldsymbol{\phi}'|}{a}\left(\mathbb{1} - \mathbf{P}^\parallel\right)\left[\mathcal{D}_\eta^2 + 2\frac{|\boldsymbol{\phi}'|'}{|\boldsymbol{\phi}'|}\mathcal{D}_\eta + \frac{|\boldsymbol{\phi}'|''}{|\boldsymbol{\phi}'|} - \mathcal{H}^2 - \mathcal{H}'\right]\boldsymbol{\delta}\mathbf{v}. \tag{181}$$

Since $\mathbf{P}^\parallel$ only gives a non-vanishing contribution if it is applied to $\mathcal{D}_\eta\boldsymbol{\delta}\mathbf{v}$ or $\mathcal{D}_\eta^2\boldsymbol{\delta}\mathbf{v}$, we consider those terms separately and find

$$\mathbf{P}^\parallel\left(\mathcal{D}_\eta^2 + 2\frac{|\boldsymbol{\phi}'|'}{|\boldsymbol{\phi}'|}\mathcal{D}_\eta\right)\boldsymbol{\delta}\mathbf{v} = -\left(\frac{(\mathcal{D}_\eta^2\boldsymbol{\phi}')^\perp}{|\boldsymbol{\phi}'|}\cdot\boldsymbol{\delta}\mathbf{v} + 2\frac{(\mathcal{D}_\eta\boldsymbol{\phi}')^\perp}{|\boldsymbol{\phi}'|}\cdot\mathcal{D}_\eta\boldsymbol{\delta}\mathbf{v}\right)\mathbf{e}_1. \tag{182}$$

To obtain this expression, we wrote $\mathbf{P}^\parallel = \mathbf{e}_1\boldsymbol{\phi}'^\dagger/|\boldsymbol{\phi}'|$ and used the two relations that are obtained by taking the first and second derivative with respect to conformal time of the equation $\boldsymbol{\phi}'\cdot\boldsymbol{\delta}\mathbf{v} = 0$:

$$\boldsymbol{\phi}'\cdot\mathcal{D}_\eta\boldsymbol{\delta}\mathbf{v} = -\mathcal{D}_\eta\boldsymbol{\phi}'\cdot\boldsymbol{\delta}\mathbf{v}, \qquad \boldsymbol{\phi}'\cdot\mathcal{D}_\eta^2\boldsymbol{\delta}\mathbf{v} = -\mathcal{D}_\eta^2\boldsymbol{\phi}'\cdot\boldsymbol{\delta}\mathbf{v} - 2(\mathcal{D}_\eta\boldsymbol{\phi}')^\perp\cdot\left(\mathcal{D}_\eta - \frac{|\boldsymbol{\phi}'|'}{|\boldsymbol{\phi}'|}\right)\boldsymbol{\delta}\mathbf{v},$$

where in the second relation we used $\mathcal{D}_\eta\boldsymbol{\phi}' = (\mathcal{D}_\eta\boldsymbol{\phi}')^\perp + \frac{|\boldsymbol{\phi}'|'}{|\boldsymbol{\phi}'|}\boldsymbol{\phi}'$ and the first relation.

Now we can insert the expressions (180), (181) and (182) into (175) and find

$$\begin{aligned}&\left[\mathcal{D}_\eta^2 + 2\frac{|\boldsymbol{\phi}'|'}{|\boldsymbol{\phi}'|}\mathcal{D}_\eta + \left(\frac{|\boldsymbol{\phi}'|''}{|\boldsymbol{\phi}'|} - \mathcal{H}' - \mathcal{H}^2\right) + a^2(\mathbf{M}^2)^{\perp\perp} - \Delta - \mathbf{R}(\boldsymbol{\phi}',\boldsymbol{\phi}')\right]\boldsymbol{\delta}\mathbf{v} \\ &+ 4\left[\frac{1}{\mathcal{H}\sqrt{2\tilde{\epsilon}}}\Delta u + \frac{(\mathcal{D}_\eta\boldsymbol{\phi}')^\perp}{|\boldsymbol{\phi}'|}\cdot\boldsymbol{\delta}\mathbf{v}\right]\frac{(\mathcal{D}_\eta\boldsymbol{\phi}')^\perp}{|\boldsymbol{\phi}'|} + \left[\frac{(\mathcal{D}_\eta^2\boldsymbol{\phi}')^\perp}{|\boldsymbol{\phi}'|}\cdot\boldsymbol{\delta}\mathbf{v} + 2\frac{(\mathcal{D}_\eta\boldsymbol{\phi}')^\perp}{|\boldsymbol{\phi}'|}\cdot\mathcal{D}_\eta\boldsymbol{\delta}\mathbf{v}\right]\mathbf{e}_1 = 0.\end{aligned} \tag{183}$$

Using (53) we obtain the final form in terms of the slow-roll functions:

$$\begin{aligned}&\left[\mathcal{D}_\eta^2 + 2\mathcal{H}\left(1 + \tilde{\eta}^\parallel\right)\mathcal{D}_\eta + \mathcal{H}^2\left(3\tilde{\eta}^\parallel + (\tilde{\eta}^\perp)^2 + \tilde{\xi}^\parallel\right) + a^2(\mathbf{M}^2)^{\perp\perp} - \Delta - \mathbf{R}(\boldsymbol{\phi}',\boldsymbol{\phi}')\right]\boldsymbol{\delta}\mathbf{v} \\ &\quad + \left[\mathcal{H}^2(3\tilde{\boldsymbol{\eta}} + \tilde{\boldsymbol{\xi}})^\perp\cdot\boldsymbol{\delta}\mathbf{v} + 2\mathcal{H}\tilde{\boldsymbol{\eta}}^\perp\cdot\mathcal{D}_\eta\boldsymbol{\delta}\mathbf{v}\right]\mathbf{e}_1 + 4\left[\mathcal{H}^2\tilde{\boldsymbol{\eta}}^\perp\cdot\boldsymbol{\delta}\mathbf{v} + \frac{1}{\sqrt{2\tilde{\epsilon}}}\Delta u\right]\tilde{\boldsymbol{\eta}}^\perp = 0,\end{aligned} \tag{184}$$

which corresponds to the $\boldsymbol{\delta}\mathbf{v}$ component of the matrix equation (61).

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